

EXPONENTIAL GROWTH OF COLORED HOMFLY-PT HOMOLOGY

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ABSTRACT. We define reduced colored \mathfrak{sl}_N link homologies and use deformation spectral sequences to characterize their dependence on color and rank. We then define reduced colored HOMFLY-PT homologies and prove that they arise as large N limits of \mathfrak{sl}_N homologies. Together, these results allow proofs of many aspects of the physically conjectured structure of the family of type A link homologies. In particular, we verify a conjecture of Gorsky, Gukov and Stošić about the growth of colored HOMFLY-PT homologies.

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1. INTRODUCTION

It is well-known that the famous *Jones polynomial* is an instance of the large class of *Reshetikhin-Turaev (RT) invariants* of knots and links [48]. These are rational functions in one variable q and depend on a datum consisting of an oriented, framed link \mathcal{L} in S^3 , a complex semisimple Lie algebra \mathfrak{g} and a labelling (coloring) of the connected components of \mathcal{L} by representations of \mathfrak{g} . The Jones polynomial arises in the case where $\mathfrak{g} = \mathfrak{sl}_2$ and all link components are labelled by the vector representation.

Given such a large family of invariants, it is natural to inquire into their dependence on the datum of link, Lie algebra and labelling. The best-studied subfamily is the one where $\mathfrak{g} \in \{\mathfrak{sl}_N\}_{N \in \mathbb{N}}$ and many

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important aspects of the relationship between these invariants have been uncovered. For example, for a fixed link and labelling, it is well-known that the \mathfrak{sl}_N invariants stabilize for large N to a *colored HOMFLY-PT* invariant in two variables a and q , which contains generalizations of the Alexander polynomial and from which the \mathfrak{sl}_N invariants can be recovered as one-variable specializations $a = q^N$. The dependence of HOMFLY-PT invariants on the labelling is governed by recurrence relations [17], which are notoriously difficult to compute and are conjectured to have deep relationships to the A-polynomial, character varieties and possibly even knot contact homology, see [16, 15, 14, 21] and references therein. However, as regards questions like the one about dependence of the invariant under variations of the input link, the RT invariants seem to carry too little structure to allow meaningful answers.

This deficit has been remedied by the development of *link homology theories* categorifying the RT invariants, starting with Khovanov’s categorification of the Jones polynomial [26]. They associate to a datum of link, Lie algebra and labelling a bi-graded vector space (usually as the homology of a graded chain complex), whose graded Euler characteristic recovers the corresponding RT invariant. Additionally, for every smooth cobordism between two links in \mathbb{R}^3 , one can construct a linear map between the corresponding invariants, and it is expected (and proven in some cases, see e.g. [25, 2, 10, 4]) that this construction is functorial and only depends on the isotopy type of the cobordism. This provides a much more satisfactory answer to the question about dependence of the invariant on the underlying link than would have been possible for the original RT invariants. Furthermore, some of these purely combinatorial link homologies can be related to gauge-theoretic link invariants, from which they inherit strong properties, for example that Khovanov homology detects the unknot [32], a question which is still open for the Jones polynomial. However, the increase in sophistication in categorified invariants is bought at the price of increased algebraic, combinatorial or geometric complexity in the definition. In fact, only for the simplest cases it is possible to compute link homologies directly from the definition. In parallel, the questions about the dependence of the invariant on the other parts of the datum become correspondingly harder and also more interesting.

The purpose of this paper is to study the rank and labelling (color) dependence of \mathfrak{sl}_N link homologies with respect to fundamental (exterior power) representations. Such link homologies have been constructed by Wu and Yonezawa [59, 60] and many others¹, building on the pioneering work of Khovanov-Rozansky [29], but up until now, little has been known about the relationships between members of this family.

1.1. The large N limit, categorified. For knots and in the *uncolored* case of the labelling by the vector representation of \mathfrak{sl}_N , Rasmussen [47] has categorified the relationship between the HOMFLY-PT polynomial and the \mathfrak{sl}_N RT polynomials by constructing *specialization spectral sequences* from Khovanov-Rozansky’s reduced triply-graded HOMFLY-PT homology [30] to their reduced \mathfrak{sl}_N homologies [29]. These spectral sequences degenerate for large N and exhibit the triply-graded homology as a stable limit of the \mathfrak{sl}_N homologies. In this sense, all uncolored reduced \mathfrak{sl}_N homologies can be packaged into a single link homology theory together with a family of specialization spectral sequences.

In this paper, we define bi-graded *reduced colored \mathfrak{sl}_N homologies* and triply-graded *reduced colored HOMFLY-PT homologies* and show that they satisfy analogous relationships.

Theorem 1. *Let $\mathcal{L}(i)$ be a labelled link with a marked component i of minimal label. There is a spectral sequence*

$$\overline{\text{KhR}}^\infty(\mathcal{L}(i)) \rightsquigarrow \overline{\text{KhR}}^{\mathfrak{sl}_N}(\mathcal{L}(i))$$

from colored HOMFLY-PT homology to colored \mathfrak{sl}_N homology, both reduced at the component i . See Theorem 3.43.

¹We refer the reader to the introduction of [37] for a brief survey of the different approaches to colored \mathfrak{sl}_N link homologies and an argument for why they are equivalent, which is based on work of Cautis [7].

Theorem 2. . Let \mathcal{K} be a labelled knot. For sufficiently large N , there are isomorphisms

$$\bigoplus_{\substack{i+2Nj=I \\ h-j=J}} \overline{\text{KhR}}_{i,j,h}^{\infty}(\mathcal{K}) \cong \overline{\text{KhR}}_{I,J}^{\mathfrak{sl}_N}(\mathcal{K})$$

between a grading-collapsed version of reduced colored HOMFLY-PT homology and reduced colored \mathfrak{sl}_N homology. See Theorem 3.46.

Theorem 2 relies on the fact that the reduced colored HOMFLY-PT homology is finite-dimensional for knots, see Proposition 3.45. On the contrary, the original *unreduced* colored HOMFLY-PT homology of Mackaay-Stošić-Vaz [36] and Webster-Williamson [53] is always infinite-dimensional. Nevertheless, we find an analogue of Theorem 1, which relates it to unreduced colored \mathfrak{sl}_N homologies and – more generally – their Σ -deformed versions, which have been defined by Wu in [58] and further studied by Rose and the author in [49].

Theorem 3. Let \mathcal{L} be a labelled link. There is a spectral sequence

$$\text{KhR}^{\infty}(\mathcal{L}) \rightsquigarrow \text{KhR}^{\Sigma}(\mathcal{L})$$

from unreduced colored HOMFLY-PT homology to unreduced, Σ -deformed colored \mathfrak{sl}_N homology. See Theorem 3.22.

1.2. The physical structure. The existence of reduced colored \mathfrak{sl}_N and HOMFLY-PT homologies and the theorems describing their relationships, although new, will not come as a surprise to the expert. In fact, they are at the very centre of current interest in the study of link homologies from the perspective of theoretical physics. Since Gukov-Schwarz-Vafa’s physical interpretation of Khovanov-Rozansky homology as a space of BPS states [22], there has been significant cross-fertilization between the mathematical and physical sides of this field. A particularly interesting outcome is a package of conjectures about the structure of the family of colored \mathfrak{sl}_N and HOMFLY-PT homologies, which has been developed over a decade in a series of papers by Dunfield-Gukov-Rasmussen [12], Gukov-Walcher [24], Gukov-Stošić [23], Gorsky-Gukov-Stošić [18] and most recently Gukov-Nawata-Saberi-Stošić-Sułkowski [20].

Thanks to recent advances in understanding the higher representation theory underlying link homologies (in particular *categorical skew Howe duality*), which led to the convergence of several different approaches to link homologies [7, 42, 49, 37], we are now in a position to prove a significant part of the physically conjectured structure. In the following theorems, \mathcal{K}^k denotes a knot labelled with the k^{th} exterior power of the vector representation.

Theorem 4. There is a spectral sequence

$$\overline{\text{KhR}}^{\infty}(\mathcal{K}^k) \rightsquigarrow \left(\overline{\text{KhR}}^{\infty}(\mathcal{K}^1) \right)^{\otimes k}$$

which preserves the homological grading. See Corollary 3.47.

This implies that the colored HOMFLY-PT homologies of a knot grow at least exponentially in color, as predicted in [18]. A related result has been obtained previously by the author for categorical invariants of rational tangles, [54]. The next theorem provides rank-reducing spectral sequences, as suggested in [12, 24]. In special cases ($M = k = 1$), these have been defined and used before by Lewark-Lobb in [33] for the construction of new slice genus bounds.

Theorem 5. For $N \geq M$ there is a spectral sequence

$$\overline{\text{KhR}}^{\mathfrak{sl}_N}(\mathcal{K}^k) \rightsquigarrow \overline{\text{KhR}}^{\mathfrak{sl}_M}(\mathcal{K}^k)$$

which preserves the homological grading. See Corollary 2.32.

We are optimistic that these and other spectral sequences (Theorem 2.30) will help to understand relations in the zoo of slice genus bounds and concordance homomorphisms obtained from \mathfrak{sl}_N link homologies [34, 57, 33] as analogons of Rasmussen's s -invariant [46]. Finally, there are color-reducing spectral sequences, analogous to the colored differentials in [23, 18].

Theorem 6. *For $k \geq h$ there is a spectral sequence*

$$\overline{\text{KhR}}^\infty(\mathcal{K}^k) \rightsquigarrow \overline{\text{KhR}}^\infty(\mathcal{K}^h)$$

which preserves the homological grading (up to a shift). See Corollary 3.48.

1.3. Open problems. While this paper solves a significant part of the physically conjectured structure from [12, 24, 23, 18, 20], several adventurous questions remain open and deserve further mathematical attention. We finish this introduction with a list of some of these open problems (roughly in increasing order of perceived difficulty) and comment on their relation to the results in the present work.

The module structure of colored HOMFLY-PT homology. In our definition of reduced colored HOMFLY-PT homology we assume that the reduction happens on a link component of minimal label. Related to this technical restriction are questions about the relationship between reduced and unreduced HOMFLY-PT homology, see Remark 3.42, and about extensions of colored \mathfrak{sl}_N and HOMFLY-PT homologies to invariants of pointed links along the lines of Baldwin-Levine-Sarkar [1]. Further, if this assumption can be avoided or circumvented, then proofs for some of the conjectures about color-shifting properties on links made in [20] should be within reach.

Are the spectral sequences actually differentials? Theorem 2 implies that the spectral sequences in Theorem 5 are degenerate for sufficiently large M . In general, it is unclear how fast the spectral sequences from Theorems 1 and 3-6 converge and whether the spectral sequences between colored HOMFLY-PT homologies respect the third a -grading. However, there are conjectures about these properties in [12, 23, 18, 20], which invite further investigation.

Reduced homologies for non-fundamental colors. Unreduced colored \mathfrak{sl}_N homologies have also been defined for labellings by non-fundamental representations via cabling operations by infinite-twist torus links, see Rozansky [50] and Cautis [7]. Provided compatible definitions of reduced colored \mathfrak{sl}_N and HOMFLY-PT homologies for these labellings, many of the results in the present paper can be expected to extend to this setting to produce specialization, rank-reducing and color-reducing spectral sequences.

The q -symmetry and another grading. A long-standing conjecture about reduced HOMFLY-PT homologies posits that there is an involution on reduced uncolored HOMFLY-PT homology, which inverts the q -grading. On reduced colored HOMFLY-PT homologies, it is expected to swap labellings with their transposes, e.g. exterior power with symmetric power representations. This q -symmetry, also called *mirror symmetry* in [12, 23, 18, 20], is further conjectured to intertwine with specialization spectral sequences and to be related to an alternative homological grading $-t_r$ in their notation – effectively making colored HOMFLY-PT homology quadruply-graded.

Lie superalgebra link homologies. Gorsky-Gukov-Stošić [18] have suggested that the q -symmetry and its related structure might be more naturally interpreted in a hypothetical framework of colored $\mathfrak{gl}_{N|M}$ link homologies, whose $M = 0$ cases are related to the known \mathfrak{sl}_N homologies. Recent work of the author in collaboration with Tubbenhauer and Vaz [51] supports this conjecture on the decategorified level by exhibiting the q -symmetry of colored HOMFLY-PT polynomials as a consequence of relations between the representation categories of quantum $\mathfrak{gl}_{N|M}$ for various parameters N, M . Some progress towards a definition of $\mathfrak{gl}_{N|M}$ link homologies via categorical skew Howe duality has been made by

Grant [19]. A string theory setup for such link homologies has been proposed by Mikhaylov-Witten [39].

A spectral sequence to knot Floer homology. The classical Alexander polynomial can be recovered as the $a = 1$ specialization of the (uncolored) HOMFLY-PT polynomial. Dunfield-Gukov-Rasmussen [12] have conjectured the existence of a corresponding specialization spectral sequence from (uncolored) HOMFLY-PT homology to knot Floer homology [41, 44]. Recent developments related to this conjecture and the present paper include work by Dowlin [11] and Baldwin-Levine-Sarkar [1], who match up structural properties of both types of link homologies, and Ellis-Petkova-Vértési [13], whose results can be interpreted as saying that knot Floer homology (and its extension to tangles) is already a $\mathfrak{gl}_{1|1}$ homology.

Outline of the paper: Theorems 1, 2 and 3 are proved in Section 3 using \mathbb{Z} -graded matrix factorizations and arguments analogous to those in Rasmussen’s work on the uncolored case [47]. The keystone Lemma 3.23 is proved using a technique inspired by the annular web evaluation algorithm of Queffelec-Rose [43]. Theorems 4, 5 and 6 follow via large N stabilization from analogous results for reduced \mathfrak{sl}_N link homologies. These are proved using the deformation theory for reduced \mathfrak{sl}_N link homologies, which is developed in Section 2, building on earlier work of the author in collaboration with Rose on the unreduced case [49] and work of Lewark-Lobb on the uncolored case [33]. For this we use the \mathfrak{sl}_N foam 2-categories introduced by Queffelec-Rose in [42], building on earlier work of Mackaay-Stošić-Vaz [35]. For the reader’s convenience we will recall matrix factorizations and foam 2-categories and explain their use in the construction of link homology theories. However, in the interest of compactness we will blackbox some technical aspects, especially regarding the relationships between the link homologies defined using foams and matrix factorization, for which we refer the reader to the cited literature.

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Conventions: All links and tangles are assumed to be oriented and framed, link and tangle diagrams are assumed to be oriented and they represent oriented, framed links and tangles via the blackboard framing.

2. DEFORMED COLORED \mathfrak{sl}_N HOMOLOGY

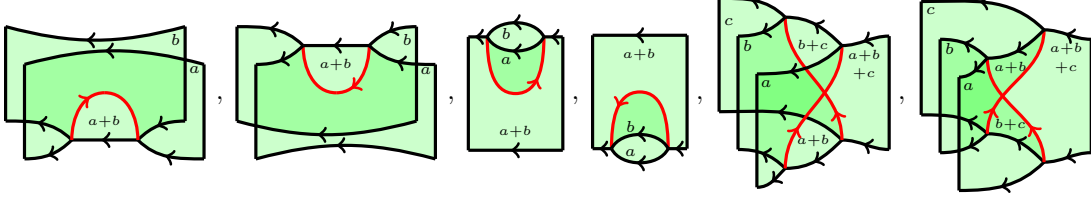
Sections 2.1 and 2.2 recall the essential features of the foam 2-categories of Queffelec-Rose [42] and their use in the construction of (deformed) colored \mathfrak{sl}_N Khovanov-Rozansky homologies. In Section 2.3 we consider these link homologies as modules over the corresponding unknot invariants. Section 2.4 extends the deformation theory of \mathfrak{sl}_N link homologies from [49] to $(1, 1)$ -tangles. This allows the definition of corresponding reduced homologies in Section 2.5 and deformation spectral sequences in Section 2.6, which leads to a proof of Theorem 5.

2.1. Foams. The 2-category $N\mathbf{Foam}$ consists of objects given by sequences $\mathbf{a} = (a_1, \dots, a_m)$ for $m \geq 0$ with $a_i \in \{1, \dots, N\}$, 1-morphisms are formal direct sums of enhanced \mathfrak{sl}_N webs – leftward oriented,

labelled trivalent graphs generated under planar composition by

$$\begin{array}{c} a \\ \swarrow \\ \text{---} \\ \searrow \\ b \end{array} \quad a+b \quad , \quad a+b \quad \begin{array}{c} \swarrow \\ \text{---} \\ \searrow \\ b \end{array} \quad a$$

which we view as mapping from the sequence determined by the labelled points on the right boundary to the one determined by the left. The 2-morphisms are matrices of \mathbb{C} -linear combinations of enhanced \mathfrak{sl}_N foams – singular cobordisms between such webs assembled from local pieces



modulo local relations and certain isotopies, for which we refer the reader to [42, Section 3.1]. The facets of these foams carry labellings by elements in $\{1, \dots, N\}$, and a k -labelled facet may also be decorated by elements from a ring of symmetric polynomials in k variables. For later use we list a family of foam relations, see [42, Relation (3.9)], which govern *decoration migration* between the three facets adjacent to a *seam* singularity:

$$(2.1) \quad \begin{array}{c} b \\ \swarrow \\ \text{---} \\ \searrow \\ a \end{array} \quad \begin{array}{c} a+b \\ \swarrow \\ \text{---} \\ \searrow \\ a \end{array} \quad \pi_\gamma = \sum_{\alpha, \beta} c_{\alpha, \beta}^\gamma \quad \begin{array}{c} b\pi_\beta \\ \swarrow \\ \text{---} \\ \searrow \\ a \end{array} \quad \begin{array}{c} a+b \\ \swarrow \\ \text{---} \\ \searrow \\ a \end{array} \quad \pi_\alpha$$

Here π_α denotes the Schur polynomial corresponding to the Young diagram α and the $c_{\alpha, \beta}^\gamma$ are the corresponding Littlewood-Richardson coefficients. Imposing this family of relations has the effect of identifying the union of the alphabets of variables on the left side facets with the alphabet on the right side facet. Analogous relations hold across all seams.

The 2-morphisms in $N\mathbf{Foam}$ admit a grading, which is first defined on undecorated foams as a certain weighted Euler characteristic, see [42, Definition 3.3], to which is added twice the sum of the degrees of the decorating symmetric polynomials in the general case. This grading is additive under disjoint union \sqcup , under composition in the 1-morphism direction \otimes and under composition in the 2-morphism direction \circ . All relations in $N\mathbf{Foam}$ are homogeneous, so this grading descends to $N\mathbf{Foam}$.

We now introduce versions of foam 2-categories used in the definition of link homologies:

- $N\mathbf{grFoam}$ is the 2-category with the same objects as $N\mathbf{Foam}$, with 1-morphisms being formal direct sums of webs as in $N\mathbf{Foam}$, but equipped with an additional formal \mathbb{Z} -grading. I.e. corresponding to a web W from $N\mathbf{Foam}$ we have webs $q^d W$ in $N\mathbf{grFoam}$, where $d \in \mathbb{Z}$ indicates the formal degree, and the automorphism shifting the grading by 1 is given by formal multiplication by q . 2-morphisms are matrices of \mathbb{C} -linear combinations of foams $F: q^{d_1} W_1 \rightarrow q^{d_2} W_2$ of grading $d_2 - d_1$.²
- $N\mathbf{Foam}^\bullet$ is the 2-category which is obtained from $N\mathbf{Foam}$ by imposing the following additional relation on 2-morphisms:

$$(2.2) \quad \begin{array}{c} 1 \\ \swarrow \\ \text{---} \\ \searrow \\ \bullet^N \end{array} = 0$$

²In [42] $N\mathbf{grFoam}$ is denoted by $N\mathbf{Foam}$.

This relation is to be understood as saying that whenever a foam contains a 1-labelled facet decorated by the N^{th} power of the facet variable (indicated by the dot), it is equal to zero. Note that this relation is homogeneous and the morphisms in $N\mathbf{Foam}^\bullet$ can be equipped with a grading. Thus it also makes sense to define:

- $N\mathbf{grFoam}^\bullet$, the 2-category which is obtained from $N\mathbf{grFoam}$ by imposing relation (2.2) on 2-morphisms.
- Let Σ be a multiset of N complex numbers. $N\mathbf{Foam}^\Sigma$ is the 2-category obtained from $N\mathbf{Foam}$ by imposing the following additional relation on 2-morphisms:

$$(2.3) \quad \begin{array}{c} 1 \\ \diagup \quad \diagdown \\ \bullet^N \end{array} = \sum_{i=0}^{N-1} (-1)^{N-i-1} e_{N-i}(\Sigma) \begin{array}{c} 1 \\ \diagup \quad \diagdown \\ \bullet^i \end{array}$$

Here $e_k(\Sigma)$ the k^{th} symmetric polynomial in N variables, evaluated at Σ . Note that, in general, this relation is not homogeneous with respect to the grading on foams. The only exception is $N\mathbf{Foam}^{\{0, \dots, 0\}} = N\mathbf{Foam}^\bullet$. If we denote by $P \in \mathbb{C}[X]$ the monic polynomial with root multiset Σ , then (2.3) can be abbreviated to $P(\bullet) = 0$.

2.2. Link homology. In this section, we recall how foam 2-categories can be used to define \mathfrak{sl}_N link homologies. For the details, the reader is referred to Queffelec-Rose [42, Section 4].

Let \mathcal{T} be a tangle diagram inside an axis-parallel rectangle in \mathbb{R}^2 , possibly with boundary on the left and right boundary of the rectangle, and with strands labelled by natural numbers. The following construction associates to the labelled tangle diagram \mathcal{T} a so-called *cube of resolutions* complex $[\![\mathcal{T}]\!]$ in the category $\mathbf{Kom}(N\mathbf{Foam})$ of chain complexes in $N\mathbf{Foam}$. In [42, Theorem 4.8] it is shown that the complex $[\![\mathcal{T}]\!]$, considered as an element of the homotopy category $\mathbf{Kom}_h(N\mathbf{Foam})$, is an invariant of the labelled tangle represented by \mathcal{T} .

Every labelled tangle admits a diagram given as the horizontal composition \otimes of tangles which are the disjoint union \sqcup of *generating tangles*: caps, cups, crossings and horizontally directed identity tangles. In fact, it suffices to define $[\![\cdot]\!]$ on such generating tangles because the operations \otimes and \sqcup on $N\mathbf{Foam}$ naturally extend to bounded complexes $\mathbf{Kom}^b(N\mathbf{Foam})$: On chain spaces they act as \sqcup and \otimes respectively and the structure of differentials is defined similarly as in a tensor product complex. Further, it is a standard result that these operations respect homotopies and, hence, are well defined on the homotopy category $\mathbf{Kom}_h^b(N\mathbf{Foam})$ of bounded complexes in $N\mathbf{Foam}$.

On 1-strand identity tangles, $[\![\cdot]\!]$ is defined as

$$[\![\leftarrow^a]\!] = \leftarrow^a, \quad [\![\rightarrow^a]\!] = \leftarrow^{N-a}.$$

and on cap and cup tangles as:

$$\begin{aligned} [\![\cap^a]\!] &= N \begin{array}{c} \leftarrow^a \\ \cap \\ \leftarrow^{N-a} \end{array}, \quad [\![\cup^a]\!] = \begin{array}{c} \leftarrow^a \\ \cup \\ \leftarrow^{N-a} \end{array} = N \\ [\![\cup^a]\!] &= N \begin{array}{c} \leftarrow^{N-a} \\ \cup \\ \leftarrow^a \end{array}, \quad [\![\cap^a]\!] = \begin{array}{c} \leftarrow^{N-a} \\ \cap \\ \leftarrow^a \end{array} = N \end{aligned}$$

Finally, for labelled, left-directed crossings one uses foam-analogues of Chuang-Rouquier's [9] Rickard complexes, see [42, Section 4.2]. For example, the complex assigned to a negative crossing of strands labelled by $a \geq b$ is

$$(2.4) \quad \left[\begin{array}{c} \leftarrow^a \\ \times \\ \leftarrow^b \end{array} \right] = \begin{array}{c} \leftarrow^a \\ \times \\ \leftarrow^b \end{array} \rightarrow q \begin{array}{c} \leftarrow^a \\ \times \\ \leftarrow^{b-1} \end{array} \rightarrow \dots \rightarrow q^b \begin{array}{c} \leftarrow^a \\ \times \\ \leftarrow^b \end{array}$$

where the underlined term is in homological degree zero, every differential is given by a single 1-facet bridging the middle *square* and we include q -degree shift that make the differentials grading preserving, allowing the complex to be defined over $N\mathbf{grFoam}$.

These assignments, together with the extension of \otimes and \sqcup to $\mathbf{Kom}^b(N\mathbf{Foam})$, define $[\![\cdot]\!]$ on all tangle diagrams.³ The resulting complex can alternatively be regarded as lying in $\mathbf{Kom}^b(N\mathbf{Foam})$ or in the graded version $\mathbf{Kom}^b(N\mathbf{grFoam})$. We use the following notation for the images of this complex under the natural quotient 2-functors:

- $[\![\mathcal{T}]\!]^{\{0,\dots,0\}} \in \mathbf{Kom}(N\mathbf{Foam}^\bullet)$ for the singly-graded undeformed version,
- $[\![\mathcal{T}]\!]^\Sigma \in \mathbf{Kom}(N\mathbf{Foam}^\Sigma)$ for the singly-graded Σ -deformed version and
- $[\![\mathcal{T}]\!]^{\mathfrak{sl}_N} \in \mathbf{Kom}(N\mathbf{grFoam}^\bullet)$ for the bi-graded undeformed version.

Theorem 2.1. [42, c.f. Theorem 4.8]. *Given a labelled tangle diagram \mathcal{T} , the complex $[\![\mathcal{T}]\!]$ in $\mathbf{Kom}(N\mathbf{Foam})$ depends, up to homotopy, only on the labelled tangle represented by \mathcal{T} . The same holds for the versions $[\![\mathcal{T}]\!]^{\{0,\dots,0\}}$, $[\![\mathcal{T}]\!]^\Sigma$ and $[\![\mathcal{T}]\!]^{\mathfrak{sl}_N}$.*

In the case that the tangle is actually a link, represented by a diagram \mathcal{L} , all of the boundary points in the complex $[\![\mathcal{L}]\!]$ are N -labelled and all webs in it are endomorphisms of a *highest weight object* of the form $\mathbf{o}^{top} := (N, \dots, N)$. Hence, one can apply the representable functor

$$\text{grtaut}: \mathbf{Kom}(\text{End}_{N\mathbf{grFoam}^\bullet}(\mathbf{o}^{top})) \rightarrow \mathbf{Kom}(\mathbf{grVect}_{\mathbb{C}})$$

given by $\text{grtaut}(\cdot) := \bigoplus_{k \in \mathbb{Z}} \text{Hom}(q^k 1_{\mathbf{o}^{top}}, \cdot)$ to $[\![\mathcal{L}]\!]^{\mathfrak{sl}_N}$ to obtain a complex of graded vector spaces. Then one defines the bi-graded vector space

$$(2.5) \quad KhR^{\mathfrak{sl}_N}(\mathcal{L}) := H_*(C^{\mathfrak{sl}_N}(\mathcal{L})) \quad \text{where } C^{\mathfrak{sl}_N}(\mathcal{L}) := \text{grtaut}([\![\mathcal{L}]\!]^{\mathfrak{sl}_N}).$$

Theorem 2.2. [42, Theorem 4.12] *Up to shifts in homological and quantum grading, $KhR^{\mathfrak{sl}_N}(\mathcal{L})$ is isomorphic as a bi-graded \mathbb{C} -vector space to Wu's and Yonezawa's colored Khovanov-Rozansky homology of the mirror link $\mathcal{L}^\#$.*⁴

In the Σ -deformed case, we can use the representable functor

$$\text{taut}: \mathbf{Kom}(\text{End}_{N\mathbf{Foam}^\Sigma}(\mathbf{o}^{top})) \rightarrow \mathbf{Kom}(\mathbf{Vect}_{\mathbb{C}})$$

given by $\text{taut}(\cdot) := \text{Hom}(1_{\mathbf{o}^{top}}, \cdot)$ to $[\![\mathcal{L}]\!]^\Sigma$ to obtain a complex of finite-dimensional vector spaces and we define the graded \mathbb{C} -vector space

$$(2.6) \quad KhR^\Sigma(\mathcal{L}) := H_*(C^\Sigma(\mathcal{L})) \quad \text{where } C^\Sigma(\mathcal{L}) := \text{taut}([\![\mathcal{L}]\!]^\Sigma).$$

Theorem 2.3. [49, Theorem 2] *Up to shifts in homological degree, $KhR^\Sigma(\mathcal{L})$ is isomorphic as a singly-graded \mathbb{C} -vector space to Wu's deformed Khovanov-Rozansky homology with respect to deformation parameters Σ .*

Remark 2.4. $KhR^{\mathfrak{sl}_N}(\mathcal{L})$ is isomorphic to $KhR^{\{0,\dots,0\}}(\mathcal{L})$ as a singly-graded \mathbb{C} -vector space.

2.3. Module structure. In this section, we show that (deformed) colored \mathfrak{sl}_N link homologies can be regarded as modules over corresponding unknot homologies. For this, let \mathcal{T} be a labelled $(1,1)$ -tangle diagram with boundary label k , i.e. a labelled tangle diagram with one right and one left endpoint, which represents a labelled link \mathcal{L} with a k -labelled component cut open.

We write $\mathbf{o}^{cut} := (N, \dots, N, k)$ for the boundary labels of objects in the complex $[\![\mathcal{T}]\!]$ and $H_k^N := H^*(Gr(k, N), \mathbb{C})$ for the (graded) cohomology ring (with coefficients in \mathbb{C}) of the Grassmannian of

³ For technical reasons, one usually assumes that all N -labelled boundary points occur at the top. This can be achieved by horizontally composing with webs of the following type: $\begin{array}{c} N \text{---} \overleftarrow{\text{---}} \text{---} a \\ \text{---} \overleftarrow{\text{---}} \text{---} N \end{array}$, $\begin{array}{c} a \text{---} \overleftarrow{\text{---}} \text{---} N \\ N \text{---} \overleftarrow{\text{---}} \text{---} a \end{array}$.

⁴In the following, we suppress this difference in conventions and always consider Wu's and Yonezawa's construction applied to the mirror link and with adjusted gradings.

complex k -planes in \mathbb{C}^N . This ring has the following presentation as the quotient of a ring of symmetric polynomials in a k -element alphabet \mathbb{X} with coefficients in \mathbb{C}

$$(2.7) \quad H_k^N \cong \frac{\text{Sym}(\mathbb{X})}{\langle h_{N-k+i}(\mathbb{X}) \mid i > 0 \rangle},$$

where $h_i(\mathbb{X})$ denotes the i^{th} complete symmetric polynomial in \mathbb{X} . We will also need the (ungraded) Σ -deformed Grassmannian cohomology ring H_k^Σ , given by the presentation

$$(2.8) \quad H_k^\Sigma := \frac{\text{Sym}(\mathbb{X})}{\langle h_{N-k+i}(\mathbb{X} - \Sigma) \mid i > 0 \rangle}$$

where the complete symmetric polynomials $h_i(\mathbb{X} - \Sigma)$ in the difference of the alphabets \mathbb{X} and Σ are given by the generating function:

$$\sum h_i(\mathbb{X} - \Sigma)t^i := \frac{\prod_{\lambda \in \Sigma}(1 - \lambda t)}{\prod_{x \in \mathbb{X}}(1 - xt)}.$$

Remark 2.5. H_k^N and H_k^Σ have (homogeneous) \mathbb{C} -bases with elements represented by the Schur polynomials $\pi_\lambda \in \text{Sym}(\mathbb{X})$ indexed by Young diagrams λ that fit into a $k \times (N - k)$ box, see e.g. [58, Theorem 2.10].

Lemma 2.6. *There are \mathbb{C} -algebra isomorphisms*

$$\text{End}_{N\mathbf{Foam}^\Sigma}(1_{\mathbf{o}^{\text{cut}}}) \cong H_k^\Sigma \quad \text{and} \quad \text{End}_{N\mathbf{Foam}^\Sigma}(1_{\mathbf{o}^{\text{top}}}) \cong \mathbb{C}$$

and graded \mathbb{C} -algebra isomorphisms

$$\text{End}_{N\mathbf{grFoam}^\bullet}(1_{\mathbf{o}^{\text{cut}}}) \cong H_k^N \quad \text{and} \quad \text{End}_{N\mathbf{grFoam}^\bullet}(1_{\mathbf{o}^{\text{top}}}) \cong \mathbb{C}.$$

Proof. In [49, Lemma 23] it was proven that

$$\text{End}_{N\mathbf{Foam}^\Sigma}(1_{(k)}) \cong H_k^\Sigma$$

where H_k^Σ can be interpreted as algebra of decorations on a k -labelled foam facet. In particular $\text{End}_{N\mathbf{Foam}^\Sigma}(1_{(0)}) \cong \mathbb{C} \cong \text{End}_{N\mathbf{Foam}^\Sigma}(1_{(N)})$. Since $\text{End}_{N\mathbf{Foam}^\Sigma}(1_{\mathbf{o}^{\text{cut}}})$ clearly contains the subalgebra H_k^Σ of decorated identity foams, it remains to show that these actually span $\text{End}_{N\mathbf{Foam}^\Sigma}(1_{\mathbf{o}^{\text{cut}}})$.

The elements of $\text{End}_{N\mathbf{Foam}^\Sigma}(1_{\mathbf{o}^{\text{cut}}})$ are given by images of closed string diagrams in categorified quantum \mathfrak{gl}_m under the appropriate foamation 2-functor, see [42, Section 3.2]. Such diagrams can always be written in terms of polynomials in non-nested bubbles, see [28, Section 3.2.1], which act as decorations on the identity foam on $1_{\mathbf{o}^{\text{cut}}}$, see [42, Equations (3.32) and (3.33)]. The undeformed, graded case is similar. \square

We define the representable functors

$$\begin{aligned} \text{grtaut}_{\text{cut}} &: \mathbf{Kom}(\text{End}_{N\mathbf{grFoam}^\bullet}(\mathbf{o}^{\text{cut}})) \rightarrow \mathbf{Kom}(H_k^N\text{-mod}) \\ \text{taut}_{\text{cut}} &: \mathbf{Kom}(\text{End}_{N\mathbf{Foam}^\Sigma}(\mathbf{o}^{\text{cut}})) \rightarrow \mathbf{Kom}(H_k^\Sigma\text{-mod}) \end{aligned}$$

given by $\text{grtaut}_{\text{cut}}(\cdot) := \bigoplus_{k \in \mathbb{Z}} \text{Hom}(q^k 1_{\mathbf{o}^{\text{cut}}}, \cdot)$ and $\text{taut}_{\text{cut}}(\cdot) := \text{Hom}(1_{\mathbf{o}^{\text{cut}}}, \cdot)$.

Definition 2.7. For $(1, 1)$ -tangle diagrams \mathcal{T} with boundary label k , the colored Khovanov-Rozansky homology and its deformed version are defined as

$$\begin{aligned} \text{KhR}^{\mathfrak{sl}_N}(\mathcal{T}) &:= H_*(C^{\mathfrak{sl}_N}(\mathcal{T})) \quad \text{where } C^{\mathfrak{sl}_N}(\mathcal{T}) := \text{grtaut}_{\text{cut}}(\llbracket \mathcal{T} \rrbracket^{\mathfrak{sl}_N}), \\ \text{KhR}^\Sigma(\mathcal{T}) &:= H_*(C^\Sigma(\mathcal{T})) \quad \text{where } C^\Sigma(\mathcal{T}) := \text{taut}_{\text{cut}}(\llbracket \mathcal{T} \rrbracket^\Sigma). \end{aligned}$$

The following is a corollary of Theorem 2.1.

Corollary 2.8. $\mathrm{KhR}^{\mathrm{sl}_N}(\cdot)$ and $\mathrm{KhR}^\Sigma(\cdot)$, as (bi-)graded vector spaces, are invariant under Reidemeister moves and their tangle analogues and, thus, define invariants of $(1,1)$ -tangles.

Lemma 2.9. If \mathcal{T} is a $(1,1)$ -tangle diagram and \mathcal{L} its closure, then there are isomorphisms $C^{\mathrm{sl}_N}(\mathcal{T}) \cong C^{\mathrm{sl}_N}(\mathcal{L})$ and $C^\Sigma(\mathcal{T}) \cong C^\Sigma(\mathcal{L})$ of complexes of (graded) vector spaces. Consequently, $\mathrm{KhR}^{\mathrm{sl}_N}(\mathcal{T}) \cong \mathrm{KhR}^{\mathrm{sl}_N}(\mathcal{L})$ and $\mathrm{KhR}^\Sigma(\mathcal{T}) \cong \mathrm{KhR}^\Sigma(\mathcal{L})$ as (bi-)graded vector spaces.

Proof. We first claim that $\llbracket \mathcal{T} \rrbracket^{\mathrm{sl}_N}$ is homotopy equivalent to a complex, whose objects are direct sums of grading shifts of a k -labelled web edge and whose differentials are matrices of decorated identity foams on this edge. The statement about the differentials follows from Lemma 2.6 once it is clear that every web appearing in the chain spaces of $\llbracket \mathcal{T} \rrbracket^{\mathrm{sl}_N}$ – regarded as a 1-term complex – is homotopy equivalent to a complex built out of grading shifted copies of the identity web $1_{\mathbf{o}^{\mathrm{cut}}}$. On the decategorified level, the main result of Cautis-Kamnitzer-Morrison [8, Theorem 3.3.1] guarantees that for any such web, there is a sequence of web relations which can be used to write this web as a linear combination of identity webs $1_{\mathbf{o}^{\mathrm{cut}}}$, see also Proposition 4.5. By construction of the foam 2-category, all of these web relations carry over⁵ to isomorphisms between 1-morphisms in the homotopy category $\mathbf{Kom}_h(\mathrm{End}_{N\mathbf{grFoaM}^\bullet}(\mathbf{o}^{\mathrm{cut}}))$, which proves the claim.

The same local computation which simplifies $\llbracket \mathcal{T} \rrbracket^{\mathrm{sl}_N}$ to a complex consisting of identity webs, will simplify $\llbracket \mathcal{L} \rrbracket^{\mathrm{sl}_N}$ to a complex whose objects are $(N-k, k)$ -bigon webs (each representing a k -labelled unknot) and whose differential is given by matrices of identity foams decorated on the k -labelled facet. In either case, after applying the representable functor $\mathrm{grtaut}_{\mathrm{cut}}$ or grtaut respectively, such a basic object contributes a direct summand isomorphic to H_k^N to the chain groups of the corresponding complex. It is immediate that the differentials agree as well after applying the representable functors. The deformed case is analogous. \square

From the definition it is clear that $C^{\mathrm{sl}_N}(\mathcal{T})$ is a complex of H_k^N -modules and $C^\Sigma(\mathcal{T})$ is a complex of H_k^Σ -modules. Thus, the same is true for their homologies. In fact, in the proof of Lemma 2.9 we have seen the following.

Corollary 2.10. $\mathrm{KhR}^{\mathrm{sl}_N}(\mathcal{L})$ is the homology of a complex of free H_k^N -modules and $\mathrm{KhR}^\Sigma(\mathcal{L})$ is the homology of a complex of free H_k^Σ -modules.

Theorem 2.11. If \mathcal{K} is a labelled knot. Then $C^{\mathrm{sl}_N}(\mathcal{K}) \in \mathbf{Kom}_h(H_k^N\text{-mod})$, $\mathrm{KhR}^{\mathrm{sl}_N}(\mathcal{K}) \in H_k^N\text{-mod}$, $C^\Sigma(\mathcal{K}) \in \mathbf{Kom}_h(H_k^\Sigma\text{-mod})$ and $\mathrm{KhR}^\Sigma(\mathcal{K}) \in H_k^\Sigma\text{-mod}$ are well-defined knot invariants, which do not depend on the particular representation of \mathcal{K} as closure of a $(1,1)$ -tangle diagram.

Proof. One-component $(1,1)$ -tangles are isotopic if and only if they close to the same knot. Then it follows from Corollary 2.8 that the module structure is also an invariant. \square

Theorem 2.11 also holds for links, although we do not use this fact here. It can be proved via the interpretation of the module structure in Section 3 and [49, Proposition 60].

Remark 2.12. The unknot homologies are given by $\mathrm{KhR}^{\mathrm{sl}_N}(\bigcirc^k) \cong H_k^N$, $\mathrm{KhR}^\Sigma(\bigcirc^k) \cong H_k^\Sigma$ and, in particular, $\mathrm{KhR}^{\{0,\dots,0\}}(\bigcirc^k) \cong H_k^{\{0,\dots,0\}}$.

2.4. Decomposing deformations. Let \mathcal{K} be a knot given as the closure of a $(1,1)$ -tangle diagram \mathcal{T} and let $\Sigma = \{\lambda_1^{N_1}, \dots, \lambda_l^{N_l}\}$ be an N -element multiset consisting of distinct complex numbers λ_j occurring with multiplicities N_j .

⁵See also the explanation in [43, Proof of Proposition 5.1]

Lemma 2.13. [49, Theorem 13] *There is an isomorphism of \mathbb{C} -algebras*

$$H_k^\Sigma \cong \bigoplus_{\sum a_j=k} \bigotimes_{j=1}^l H_{a_j}^{N_j}$$

and the direct summands are local and indexed by k -element multisubsets $A = \{\lambda_1^{a_1}, \dots, \lambda_l^{a_l}\}$ of Σ .

Definition 2.14. Let A be a k -element multisubset of Σ and denote by 1_A the corresponding idempotent in H_k^Σ , then we define the A -localized complex

$$C_A^\Sigma(\mathcal{T}) := 1_A \text{taut}_{\text{cut}}(\llbracket \mathcal{T} \rrbracket^\Sigma)$$

which is a direct summand of $C^\Sigma(\mathcal{T})$ and clearly itself a knot invariant in $\mathbf{Kom}_h(H_k^\Sigma\text{-mod})$. Then the A -localized Σ -deformed Khovanov-Rozansky homology of \mathcal{K} is defined to be

$$\text{KhR}_A^\Sigma(\mathcal{K}) := H_*(C_A^\Sigma(\mathcal{T})).$$

Note that $\text{KhR}^\Sigma(\mathcal{K}) \cong \bigoplus_{\text{multisubsets } A} \text{KhR}_A^\Sigma(\mathcal{K})$.

Theorem 2.15. Let \mathcal{K} be a knot and write \mathcal{K}^k for its k -labelled version if $k \in \mathbb{N}$. Let $A = \{\lambda_1^{a_1}, \dots, \lambda_l^{a_l}\}$ be a k -element multisubset of Σ . There is an isomorphism of singly-graded vector spaces:

$$(2.9) \quad \text{KhR}_A^\Sigma(\mathcal{K}^k) \cong \bigotimes_{j=1}^l \text{KhR}^{\text{sl}_{N_j}}(\mathcal{K}^{a_j})$$

Proof. The proof is very similar to the proof of [49, Theorem 1]. We sketch it here because we need a variation in the next section, but readers interested in the details are referred to [49].

The proof uses a slightly enlarged 2-category $N\widehat{\mathbf{Foam}}^\Sigma$, in which idempotent-decorated identity foams are split idempotents – in other words, webs are allowed to be colored by idempotents as well. Then we have that $C_A^\Sigma(\mathcal{T}) = 1_A \text{taut}_{\text{cut}}(\llbracket \mathcal{T} \rrbracket^\Sigma \cong \text{taut}_{\text{cut}}(\llbracket \mathcal{T}_A \rrbracket^\Sigma)$, where $\llbracket \mathcal{T}_A \rrbracket^\Sigma$ is the complex in $N\widehat{\mathbf{Foam}}^\Sigma$ which is associated to the $(1,1)$ -tangle diagram \mathcal{T} , with the boundary k -labelled web edges colored by the idempotent corresponding to the multisubset A .⁶

In [49, Definition 46 and Theorem 65] we (that is Rose and the author) define web splitting functors φ , which can be used to split an idempotent-colored complex in $N\widehat{\mathbf{Foam}}^\Sigma$ into its root-colored components:

$$\varphi(\llbracket \mathcal{T}_A \rrbracket^\Sigma) = \bigotimes_{j=1}^l \llbracket \mathcal{T}_{\lambda_j \in A} \rrbracket^\Sigma$$

Here we use the shorthand $\llbracket \mathcal{T}_{\lambda_j \in A} \rrbracket^\Sigma := \llbracket \mathcal{T}_{A_j}^{a_j} \rrbracket^\Sigma$ for the complex in $N\widehat{\mathbf{Foam}}^\Sigma$ assigned to the a_j -labelled tangle \mathcal{T} , with the boundary web edges labelled by $A_j := \{\lambda_j^{a_j}\}$. Furthermore, there exist representable functors $\text{taut}_{\text{split}}$ and taut_{λ_j} such that

$$(2.10) \quad \text{taut}_{\text{cut}}(\llbracket \mathcal{T}_A \rrbracket^\Sigma) \cong \text{taut}_{\text{split}}(\varphi(\llbracket \mathcal{T}_A \rrbracket^\Sigma)) \cong \bigotimes_{j=1}^l \text{taut}_{\lambda_j}(\llbracket \mathcal{T}_{\lambda_j \in A} \rrbracket^\Sigma).$$

Finally, the complexes $\llbracket \mathcal{T}_{\lambda_j \in A} \rrbracket^\Sigma$ in the tensor factors on the right-hand side of (2.10) live in 2-subcategories $N\widehat{\mathbf{Foam}}^{\lambda_j \in \Sigma}$ of $N\widehat{\mathbf{Foam}}^\Sigma$, which consist of webs and foams that are colored by multisubsets that only contain λ_j . In [49, Proposition 68] it is shown that $N\widehat{\mathbf{Foam}}^{\lambda_j \in \Sigma}$ is isomorphic to the undeformed foam 2-category $N_j\mathbf{Foam}^\bullet$. Moreover, under this isomorphism, the complex $\llbracket \mathcal{T}_{\lambda_j \in A} \rrbracket^\Sigma$ corresponds to the undeformed complex $\llbracket \mathcal{T}^{a_j} \rrbracket^{\{0, \dots, 0\}}$ and, hence, taut_{λ_j} applied to $\llbracket \mathcal{T}_{\lambda_j \in A} \rrbracket^\Sigma$ produces

⁶From now on we will abbreviate this and say “colored by A ” instead.

a complex of (ungraded) vector spaces isomorphic to the complex computing $\mathrm{KhR}^{\mathfrak{sl}_{N_j}}(\mathcal{K}^{a_j})$ with the q -grading forgotten. \square

A version of Theorem 2.15 for links with a marked component can be proved analogously, but we will not need it here.

2.5. Reduced colored \mathfrak{sl}_N homology. The purpose of this section is to define reduced colored \mathfrak{sl}_N homologies and to study their Σ -deformed versions, which depend on the choice of a k -element multiset $A \in \Sigma$. For the sake of brevity, we only consider knots \mathcal{K} , whose k -labelled versions we denote by \mathcal{K}^k . However, all definitions and results in this section have straightforward generalizations for links with a marked component.

Definition 2.16. Consider the presentation (2.7) of H_k^N and the highest degree element $\pi_k^N := e_k(\mathbb{X})^{N-k}$ in H_k^N . Then $\langle \pi_k^N \rangle$ is a 1-dimensional ideal⁷ in H_k^N and the *reduced colored \mathfrak{sl}_N Khovanov-Rozansky homology* of a k -labelled knot \mathcal{K}^k (given as the closure of a k -labelled $(1, 1)$ -tangle \mathcal{T}) is defined as

$$\overline{\mathrm{KhR}}^{\mathfrak{sl}_N}(\mathcal{K}^k) := H_*(\overline{C}^{\mathfrak{sl}_N}(\mathcal{T})) \quad \text{where } \overline{C}^{\mathfrak{sl}_N}(\mathcal{T}) := \langle \pi_k^N \rangle \mathrm{grtaut}_{\mathrm{cut}}(\llbracket \mathcal{T} \rrbracket^{\mathfrak{sl}_N}).$$

This produces bi-graded knot invariants, which take the value \mathbb{C} on the unknot.

Definition 2.17. Let A be a k -element multisubset of Σ . We define the symmetric polynomial

$$\pi_A := \prod_{\lambda \in \Sigma \setminus A} \prod_{x \in \mathbb{X}} (\lambda - x)$$

and denote by the same symbol the corresponding element in H_k^Σ .

Lemma 2.18. $\langle \pi_A \rangle$ is a 1-dimensional ideal in H_k^Σ and all such are of the form $\langle \pi_A \rangle$ for some k -element multisubset A of Σ . Moreover, the top degree homogeneous component of π_A as a symmetric polynomial in \mathbb{X} is $\pm \pi_k^N$.

Proof. First of all, note that π_A has the expansion

$$\pi_A = \prod_{\lambda \in \Sigma \setminus A} (\lambda^k - \lambda^{k-1} e_1(\mathbb{X}) + \cdots + (-1)^{k-1} \lambda e_{k-1}(\mathbb{X}) + (-1)^k e_k(\mathbb{X})) = \pm e_k(\mathbb{X})^{N-k} + \text{lower order terms}.$$

The leading term $\pm e_k(\mathbb{X})^{N-k}$ of π_A is (up to a sign) the Schur polynomial corresponding to the maximal $k \times (N - k)$ -box Young diagram. In particular, π_A is not zero in H_k^Σ , see Remark 2.5.

Checking 1-dimensionality of $\langle \pi_A \rangle$ is equivalent to showing that π_A is a simultaneous eigenvector for the $e_i(\mathbb{X})$: For a multiset of complex numbers S and the variable set \mathbb{X} , we consider the ideal $I_S := \langle h_{|S| - |X| + i}(\mathbb{X} - S) \mid i > 0 \rangle \subset \mathrm{Sym}(\mathbb{X})$. It is not hard to check that $\prod_{x \in \mathbb{X}} (\lambda - x) I_S \subset I_{S \cup \{\lambda\}}$ for any $\lambda \in \mathbb{C}$. Then, by induction, we also have $\pi_A I_A \subset I_\Sigma$ and since I_A contains all polynomials of the form $e_i(\mathbb{X}) - e_i(A)$, it follows that π_A is a $e_i(A)$ eigenvector for $e_i(\mathbb{X})$ in H_k^Σ . Also, it shows that $\pi_A \in 1_A H_k^\Sigma$ because π_A is annihilated by every $e_i(\mathbb{X}) - e_i(A)$, at least one of which would need to act as a unit in any other summand $1_B H_k^\Sigma$ by Lemma 2.19.

Finally, for every 1-dimensional ideal $I = \mathbb{C}\pi$, the homomorphism $w_I: H_k^\Sigma \rightarrow \mathbb{C}$ given by $x\pi = w_I(x)\pi$ has as kernel a maximal ideal in H_k^Σ , of which there are exactly as many as local direct summands of H_k^Σ , which have been identified as $1_A H_k^\Sigma$. Thus, we have $I = \langle \pi_A \rangle$ for some k -element multisubset A of Σ . \square

Lemma 2.19. [49, Corollary 18] *With respect to the presentation (2.8) of H_k^Σ , a symmetric polynomial $p \in \mathrm{Sym}(\mathbb{X})$ projects to a unit in the summand $1_A H_k^\Sigma$ if and only if $p(A) \neq 0$.*

⁷This is proven more generally in Lemma 2.18

Remark 2.20. For a multisubset $A \subset \Sigma$, define $\bar{A} := \{\lambda \in \Sigma \mid \lambda \text{ appears at least once in } A\}$ and $\bar{\pi}_A := \prod_{\lambda \in \bar{A} \setminus A} \prod_{x \in \mathbb{X}} (\lambda - x)$. Then $\langle \pi_A \rangle = \langle \bar{\pi}_A 1_A \rangle$ in H_k^Σ . This is easily seen, since the additional factors in the definition of π_A are units in $1_A H_k^\Sigma$ by Lemma 2.19.

Definition 2.21. Let A be a k -element multisubset of Σ . We define the A -reduced Σ -deformed Khovanov-Rozansky homology

$$\overline{\text{KhR}}_A^\Sigma(\mathcal{K}) := H_*(\overline{C}_A^\Sigma(\mathcal{T})) \quad \text{where } \overline{C}_A^\Sigma(\mathcal{T}) := \langle \pi_A \rangle \text{taut}_{\text{cut}}[\mathcal{T}]^\Sigma.$$

By Theorem 2.11 this is a singly-graded knot invariant. It takes value \mathbb{C} on the unknot.

In the uncolored case, where $A = \{\lambda\}$, these reduced deformed homologies have been previously defined by Lewark-Lobb [33], who also show that they lead to new obstructions to sliceness.

Theorem 2.22. Let $\Sigma = \{\lambda_1^{N_1}, \dots, \lambda_l^{N_l}\}$ be an N -element multiset of distinct complex numbers λ_j , occurring with multiplicities N_j . Let $A = \{\lambda_1^{a_1}, \dots, \lambda_l^{a_l}\}$ be a k -element multisubset of Σ . Then there is an isomorphism of singly-graded vector spaces:

$$(2.11) \quad \overline{\text{KhR}}_A^\Sigma(\mathcal{K}^k) \cong \bigotimes_{j=1}^l \overline{\text{KhR}}^{\text{sl}_{N_j}}(\mathcal{K}^{a_j})$$

Proof. The reduced deformed homology is computed by the chain complex $\langle \pi_A \rangle \text{taut}_{\text{cut}}[\mathcal{T}]^\Sigma$, which we interpret as a chain complex in which the objects are foams $F: 1_{\mathbf{o}^{\text{cut}}} \rightarrow W$, which are decorated by π_A on the k -facet adjacent to the domain web $1_{\mathbf{o}^{\text{cut}}}$, and W is a web appearing in the cube of resolutions chain complex $[\mathcal{T}]^\Sigma$. The differentials in $\langle \pi_A \rangle \text{taut}_{\text{cut}}[\mathcal{T}]^\Sigma$ are given by stacking foams $G: W \rightarrow W'$ from $[\mathcal{T}]^\Sigma$ on top of object foams like F .

The next step is to apply a splitting functor φ to all webs in the complex to separate out individual roots. This works exactly as in the proof of Theorem 2.15, except that the decoration by π_A on the k -facet needs to be distributed onto the l parallel a_j -facets colored by $A_j := \{\lambda_j^{a_j}\}$. We claim that if an A -colored facet is split into facets colored by multisubsets $A = B \uplus C$ which are disjoint $B \cap C = 0$, then π_A splits into the product of decorations $u_B \pi_B$ and $u_C \pi_C$ on the B - and C -colored facets respectively, where u_B and u_C are units.

To see this, denote the alphabet on the A -, B - and C -colored facets with \mathbb{X} , \mathbb{X}_1 and \mathbb{X}_2 respectively. Then the decoration migration relations (2.1) impose $\mathbb{X} = \mathbb{X}_1 \uplus \mathbb{X}_2$ and we get:

$$\pi_B \pi_C = \prod_{\lambda \in \Sigma \setminus B} \prod_{x \in \mathbb{X}_1} (\lambda - x) \prod_{\lambda \in \Sigma \setminus C} \prod_{x \in \mathbb{X}_2} (\lambda - x) = \underbrace{\prod_{\lambda \in \Sigma \setminus C} \prod_{x \in \mathbb{X}} (\lambda - x)}_{=\pi_A} \underbrace{\prod_{\lambda \in C} \prod_{x \in \mathbb{X}_1} (\lambda - x)}_{=:u_B^{-1}} \underbrace{\prod_{\lambda \in B} \prod_{x \in \mathbb{X}_2} (\lambda - x)}_{=:u_C^{-1}}$$

Using the criterion from Lemma 2.19 it is clear that the elements which we have suggestively denoted by u_B^{-1} and u_C^{-1} are indeed units. The following equation illustrates the action of the foam splitting functor φ in a simple case for the reader familiar with the notation and coloring conventions in [49].

$$(2.12) \quad \varphi \left(\begin{array}{c} \text{green square} \\ \text{dashed line } \pi_A \\ \text{arrows on edges} \end{array} \right) = \sum_r \left(\begin{array}{c} \text{blue square } g_r \\ \text{red square } f_r \\ \text{red circle } \pi_A \end{array} \right) = \left(\begin{array}{c} \text{blue square } u_B \pi_B \\ \text{red square } u_C \pi_C \end{array} \right).$$

If A contains more than two distinct roots, then an inductive argument allows to split A -facets into parallels of A_j -facets and to distribute π_A into decorations by π_{A_j} (up to units). This argument provides the necessary modification of (2.10):

$$(2.13) \quad \langle \pi_A \rangle \text{taut}_{cut}(\llbracket \mathcal{T}_A \rrbracket^\Sigma) \cong \bigotimes_{j=1}^l \langle \pi_{A_j} \rangle \text{taut}_{\lambda_j}(\llbracket \mathcal{T}_{\lambda_j \in A} \rrbracket^\Sigma).$$

Next, note that $\langle \pi_{A_j} \rangle = \langle \bar{\pi}_{A_j} \rangle$ is the unique maximal ideal in $H_{a_j}^{\Sigma_j}$, where $\Sigma_j = \{\lambda_j^{N_j}\}$. Under the isomorphism of $N\mathbf{Foam}^{\lambda_j \in \Sigma}$ with the undeformed foam 2-categories $N_j\mathbf{Foam}^\bullet$, this ideal corresponds to the maximal ideal $\langle \pi_{a_j}^{N_j} \rangle$ in $H_{a_j}^{N_j}$, which is used in the definition of reduced, undeformed a_j -colored \mathfrak{sl}_{N_j} homology. Thus, the tensor factors on the right-hand side of (2.13) compute knot homologies isomorphic to the singly-graded versions of $\overline{\text{KhR}}^{\mathfrak{sl}_{N_j}}(\mathcal{K}^{a_j})$. \square

2.6. Deformation spectral sequences. In this section we study Wu's quantum filtration on deformed colored Khovanov-Rozansky homology and its induced deformation spectral sequence [56, 58], and introduce versions for reduced colored Khovanov-Rozansky homology. We again restrict to the case of knots and leave the straightforward generalization to the case of links to the reader.

Definition 2.23. A \mathbb{C} -vector space V is called *filtered* if it is equipped with a sequence of vector subspaces $\{\mathcal{F}^p V\}_{p \in \mathbb{Z}}$ – the *filtration* – satisfying

$$\dots \subset \mathcal{F}^p V \subset \mathcal{F}^{p+1} V \subset \mathcal{F}^{p+2} V \subset \dots \quad , \quad \bigcap_{p \in \mathbb{Z}} \mathcal{F}^p V = \emptyset \quad , \quad \bigcup_{p \in \mathbb{Z}} \mathcal{F}^p V = V.$$

A \mathbb{C} -algebra A is *filtered* if its underlying \mathbb{C} -vector space is filtered and the multiplication satisfies

$$\mathcal{F}^p A \cdot \mathcal{F}^q A \subset \mathcal{F}^{p+q} A \text{ for any } p, q \in \mathbb{Z}.$$

A module M over a graded, commutative \mathbb{C} -algebra A is *filtered* if its underlying \mathbb{C} -vector space is filtered and the A -action satisfies

$$x \cdot \mathcal{F}^p M \subset \mathcal{F}^{p+\deg x} M \text{ for any } x \in A, p \in \mathbb{Z}.$$

From a filtered \mathbb{C} -algebra or an A -module X we can construct the *associated graded* object

$$\text{Gr}(X) = \bigoplus_{p \in \mathbb{Z}} \mathcal{F}^p X / \mathcal{F}^{p-1} X$$

which inherits the structure of a graded \mathbb{C} -algebra or a graded A -module respectively. We use powers of q to indicate shifts in filtration:

$$\mathcal{F}^p(q^d M) := \mathcal{F}^{p+d} M.$$

An A -module homomorphism $f: M \rightarrow N$ between filtered A -modules is *filtered* if

$$f(\mathcal{F}^p M) \subset \mathcal{F}^p N.$$

A chain complex of filtered A -modules is *filtered* if the differential is filtered. The homology of such a chain complex is again a filtered A -module and its associated graded module can be computed via a spectral sequence induced by the filtration on the chain complex, see e.g. [38, Theorem 2.6].

Example 2.24. H_k^Σ is a filtered \mathbb{C} -algebra and a filtered $\text{Sym}(\mathbb{X})$ -module with respect to the filtration

$$\mathcal{F}^p H_k^\Sigma := \{x \mid x \text{ has a representative in } \text{Sym}(\mathbb{X}) \text{ of degree } \leq p\}$$

defined via the presentation (2.8). In fact, $H_k^\Sigma \cong \text{End}_{N\mathbf{Foam}^\Sigma}(1_{\mathbf{o}^{cut}})$ is also a filtered module over the larger algebra $R := \text{Sym}(\mathbb{X} \mid \mathbb{X}_1 \mid \dots \mid \mathbb{X}_s)$ of polynomials separately symmetric in the k -element alphabet \mathbb{X} and in N -element alphabets \mathbb{X}_i , one for each occurrence of N in \mathbf{o}^{cut} . The action can be interpreted as placing additional decorations on (decorated) identity foams on $1_{\mathbf{o}^{cut}}$.

Lemma 2.25. *The associated graded algebra of H_k^Σ with respect to the filtration introduced in Example 2.24 is $\text{Gr}(H_k^\Sigma) \cong H_k^N$.*

Proof. Consider the algebra $\text{Sym}(\mathbb{X})$ and let $I_0 := \langle h_{N-k+i}(\mathbb{X}) | i > 0 \rangle$ and $I_\Sigma := \langle h_{N-k+i}(\mathbb{X} - \Sigma) | i > 0 \rangle$ be the ideals used in the presentations (2.7) and (2.8) of H_k^N and H_k^Σ respectively. I_0 is a homogeneous ideal, the quotient algebra H_k^N is graded, and so it is canonically isomorphic to its own associated graded $H_k^N \cong \text{Gr}(H_k^N)$. We shall show $\text{Gr}(H_k^N) \cong \text{Gr}(H_k^\Sigma)$.

Let I be any ideal of $\text{Sym}(\mathbb{X})$, then an elementary computation identifies the degree p part of the associated graded of $\text{Sym}(\mathbb{X})/I$ as:

$$(2.14) \quad \text{Gr}(\text{Sym}(\mathbb{X})/I)_p \cong \frac{\mathcal{F}^p \text{Sym}(\mathbb{X})}{\mathcal{F}^{p-1} \text{Sym}(\mathbb{X}) + \mathcal{F}^p \text{Sym}(\mathbb{X}) \cap I}$$

Now it is sufficient to note that in the case of the ideals I_0 and I_Σ , the corresponding denominators are equal (hint: Remark 2.5 or [58, Theorem 2.10]). Then (2.14) shows that the graded parts of $\text{Gr}(H_k^N)$ and $\text{Gr}(H_k^\Sigma)$ are canonically isomorphic and that the algebra structures agree since they are both inherited from $\text{Sym}(\mathbb{X})$. \square

The existence of deformation spectral sequences for colored Khovanov-Rozansky homology has been established via Theorems 2.26 and 2.27, which are due to Wu.

Theorem 2.26. [58, Theorem 9.3] *Let \mathcal{K} be a labelled knot diagram. $C^\Sigma(\mathcal{K})$ carries a quantum filtration and the filtered homotopy type of $C^\Sigma(\mathcal{K})$ is a knot invariant.*

Theorem 2.27. [58, Theorem 9.9] *Let \mathcal{K} be a labelled knot and Σ an N -element multiset of complex numbers. Then the deformation spectral sequence induced by the quantum filtration on $C^\Sigma(\mathcal{K})$ satisfies:*

- the E_0 page is isomorphic to the undeformed bi-graded Khovanov-Rozansky complex $C^{\text{sl}_N}(\mathcal{K})$
- the E_1 page is isomorphic to the undeformed bi-graded Khovanov-Rozansky homology $\text{KhR}^{\text{sl}_N}(\mathcal{K})$
- the E_∞ page is isomorphic as a bi-graded \mathbb{C} -vector space to the associated graded of the Σ -deformed Khovanov-Rozansky homology:

$$E_\infty \cong \text{Gr}(\text{KhR}^\Sigma(\mathcal{K}))$$

and as a singly-graded \mathbb{C} -vector space, it is isomorphic to $\text{KhR}^\Sigma(\mathcal{K})$ itself.

Convention 2.28. *Given a spectral sequence as in Theorem 2.27, we abuse notation and say there is a spectral sequence: $\text{KhR}^{\text{sl}_N}(\mathcal{K}) \rightsquigarrow \text{KhR}^\Sigma(\mathcal{K})$.*

We will now give an explicit description of the quantum filtration in the framework of foams and a sketch proof of Theorem 2.27.

Sketch proof of Theorem 2.27. We work with a $(1,1)$ -tangle diagram \mathcal{T} whose closure is \mathcal{K} . This has the advantage of allowing an analogous proof of Theorem 2.30.

The original cube of resolutions complex $[\![\mathcal{T}]\!]$ can be regarded as living in $N\mathbf{grFoam}$, i.e. it is q -graded with grading preserving differentials. Analogously as in the proof of Lemma 2.9, this complex can be simplified to a (graded) homotopy equivalent complex, which consists only of identity webs $1_{\mathbf{out}}$, and decorated, grading preserving identity foams between them⁸ – by abuse of notation we also denote this by $[\![\mathcal{T}]\!]$. Next we consider the corresponding complex in $N\mathbf{Foam}$, but we remember the q -grading shifts of the objects in the complex. Proceeding to the quotient $[\![\mathcal{T}]\!]^\Sigma$ in $N\mathbf{Foam}^\Sigma$ has the effect of equating decorations by high-degree symmetric polynomials on k -facets with decorations by symmetric polynomials of lower degree and specializing decorations on N -facets to multiplication by complex

⁸The main difference here is that N -labelled foam facets can carry decorations, which can be regarded as “equivariant” parameters in the sense of [31] and [58].

scalars. Thus, the formal degree of any foam $F: q^{d_1} 1_{\mathbf{o}^{cut}} \rightarrow q^{d_2} 1_{\mathbf{o}^{cut}}$ appearing in a differential in the new complex $[\mathcal{T}]^\Sigma$ is actually less than or equal to $d_2 - d_1$. When applying the functor taut_{cut} to $[\mathcal{T}]^\Sigma$ every $q^{d_1} 1_{\mathbf{o}^{cut}}$ in the complex will contribute a copy of deformed Grassmannian cohomology ring to the chain groups of $C^\Sigma(\mathcal{T})$ according to Lemma 2.6. Now, however, we consider this copy as being equipped with the shifted filtration $q^d H_k^\Sigma$ from Example 2.24. This defines filtrations on all chain groups of $C^\Sigma(\mathcal{T})$ and we will check that the differential is filtered too. An element of $\mathcal{F}^p \text{taut}_{cut}(q^{d_1} 1_{\mathbf{o}^{cut}})$ is given by an identity foam decorated by a polynomial of degree $\leq p + d_1$. By the above discussion, applying the differential $\text{taut}_{cut}(F)$ adds a further decoration by a symmetric polynomial of degree $\leq d_2 - d_1$. The resulting total decoration on the identity foam is of degree $\leq p + d_2$, so it lies in $\mathcal{F}^p \text{taut}_{cut}(q^{d_2} 1_{\mathbf{o}^{cut}})$.

Now we know that $C^\Sigma(\mathcal{T})$ is endowed with a filtration, and since it is clearly bounded, there is an induced spectral sequence which converges to $\text{KhR}^\Sigma(\mathcal{K})$. Finally, we argue that $E_0 := \text{Gr}(C^\Sigma(\mathcal{T}))$ is isomorphic to $C^{\text{sl}_N}(\mathcal{T})$. For the chain groups, this follows from the fact $\text{Gr}(H_k^\Sigma) \cong H_k^N$, which was observed in Lemma 2.25. Moreover, it is clear from the above discussion that taking the grading preserving (top degree) part of the differential in $C^\Sigma(\mathcal{T})$ recovers the differential in $C^{\text{sl}_N}(\mathcal{T})$. The statement about the E_1 page follows immediately. \square

Remark 2.29. Theorem 2.27 also holds for links \mathcal{L} , although in this case an analogous proof using closed diagrams (and endomorphism categories of \mathbf{o}^{top} instead of \mathbf{o}^{cut}) would appear to be more natural.

We will need deformation spectral sequences for reduced colored Khovanov-Rozansky homology.

Theorem 2.30. *Let Σ be an N -element multiset of complex numbers and A a k -element multisubset of Σ . Then there is a reduced deformation spectral sequence satisfying the following conditions:*

- the E_0 page is isomorphic to the reduced bi-graded Khovanov-Rozansky complex $\overline{C}(\mathcal{K}^k)^{\text{sl}_N}$
- the E_1 page is isomorphic to the reduced bi-graded Khovanov-Rozansky homology $\overline{\text{KhR}}^{\text{sl}_N}(\mathcal{K}^k)$
- the E_∞ page is isomorphic to the associated graded of the A -reduced Σ -deformed Khovanov-Rozansky homology, which by Theorem 2.22 decomposes into a tensor product of undeformed reduced homologies of smaller color and rank:

$$E_\infty \cong \text{Gr}\left(\overline{\text{KhR}}_A^\Sigma(\mathcal{K}^k)\right) \stackrel{\text{Theorem 2.22}}{\cong} \text{Gr}\left(\bigotimes_{j=1}^l \overline{\text{KhR}}^{\text{sl}_{N_j}}(\mathcal{K}^{a_j})\right)$$

if $\Sigma = \{\lambda_1^{N_1}, \dots, \lambda_l^{N_l}\}$ and $A = \{\lambda_1^{a_1}, \dots, \lambda_l^{a_l}\}$. As a singly-graded \mathbb{C} -vector space, the E_∞ page is isomorphic to $\overline{\text{KhR}}_A^\Sigma(\mathcal{K}^k) \cong \bigotimes_{j=1}^l \overline{\text{KhR}}^{\text{sl}_{N_j}}(\mathcal{K}^{a_j})$ itself.

Proof. As before, we work with a $(1, 1)$ -tangle \mathcal{T} instead of \mathcal{K} . It is clear that $\overline{C}_A^\Sigma(\mathcal{T}) := \langle \pi_A \rangle \text{taut}_{cut}[\mathcal{T}]^\Sigma$ inherits a quantum filtration from $\text{taut}_{cut}[\mathcal{T}]^\Sigma$ and the induced spectral sequence computes $\overline{\text{KhR}}_A^\Sigma(\mathcal{K}^k)$. Moreover, the differentials in the complexes $\overline{C}_A^\Sigma(\mathcal{T})$ and $C^\Sigma(\mathcal{T})$ are given by composition by the same foams. Thus, it only remains to show that the associated graded of a chain group of $\overline{C}_A^\Sigma(\mathcal{T})$ is isomorphic to the corresponding chain group of $\overline{C}(\mathcal{K}^k)^{\text{sl}_N}$. Again, it suffices to check this for the elementary building blocks of the chain groups:

$$\text{Gr}(\langle \pi_A \rangle \text{taut}_{cut}(1_{\mathbf{o}^{cut}})) \cong \text{Gr}(\langle \pi_A \rangle H_k^\Sigma) \cong \langle \pi_k^N \rangle H_k^N \cong \langle \pi_k^N \rangle \text{grtaut}(1_{\mathbf{o}^{cut}})$$

The middle isomorphism follows from Example 2.24, Lemma 2.25 and Lemma 2.18. \square

Corollary 2.31. *There exist spectral sequences*

$$(2.15) \quad \overline{\text{KhR}}^{\mathfrak{sl}_{kN}}(\mathcal{K}^k) \rightsquigarrow \left(\overline{\text{KhR}}^{\mathfrak{sl}_N}(\mathcal{K}^1) \right)^{\otimes k} \quad \text{and}$$

$$(2.16) \quad \overline{\text{KhR}}^{\mathfrak{sl}_{2N}}(\mathcal{K}^k) \rightsquigarrow \overline{\text{KhR}}^{\mathfrak{sl}_N}(\mathcal{K}^{k-1}) \otimes \overline{\text{KhR}}^{\mathfrak{sl}_N}(\mathcal{K}^1).$$

Proof. By Theorem 2.30 there is a spectral sequence from $\overline{\text{KhR}}^{\mathfrak{sl}_N}(\mathcal{K}^k)$ to $\overline{\text{KhR}}_A^\Sigma(\mathcal{K}^k)$ for any choice of k -element multisubset A of an N -element multiset Σ of complex deformation parameters. For the first type of spectral sequence we choose Σ and A such that $l = k$, $N_1 = \dots = N_l = N$ and $a_1 = \dots = a_l = 1$ and for the second type we choose $l = 2$, $N_1 = N_2 = N$ and $a_1 = k - 1$, $a_2 = 1$. \square

Corollary 2.32. (Rank-reducing spectral sequences) *For $N \geq M$ there is a spectral sequence*

$$(2.17) \quad \overline{\text{KhR}}^{\mathfrak{sl}_N}(\mathcal{K}^k) \rightsquigarrow \overline{\text{KhR}}^{\mathfrak{sl}_M}(\mathcal{K}^k).$$

Proof. Choose $l = 2$ and $N_1 = M$, $N_2 = N - M$ and $a_1 = k$, $a_2 = 0$ in Theorem 2.30 and note that $\overline{\text{KhR}}^{\mathfrak{sl}_{N-M}}(\mathcal{K}^0) \cong \mathbb{C}$. \square

Corollary 2.33. *For $k \geq h$ there is a spectral sequence*

$$(2.18) \quad \overline{\text{KhR}}^{\mathfrak{sl}_{N+k-h}}(\mathcal{K}^k) \rightsquigarrow \overline{\text{KhR}}^{\mathfrak{sl}_N}(\mathcal{K}^h).$$

Proof. Choose $l = 2$ and $N_1 = N$, $N_2 = k - h$ and $a_1 = h$, $a_2 = k - h$ in Theorem 2.30 and note that $\overline{\text{KhR}}^{\mathfrak{sl}_{k-h}}(\mathcal{K}^{k-h}) \cong \mathbb{C}$. \square

Remark 2.34. In general, it is unclear how fast the spectral sequences in the previous corollaries converge. However, it follows from the stability property of Theorem 2 in the introduction (see also Theorem 3.46) that the spectral sequences from Corollary 2.32 are degenerate for large M .

For completeness we mention the following theorem, which follows from Corollary 2.10 similarly as in the uncolored case, see e.g. Rasmussen's [45, Lemma 5.5].

Theorem 2.35. *There is an unreducing spectral sequence*

$$\text{KhR}^{\mathfrak{sl}_N}(\bigcirc^k) \otimes \overline{\text{KhR}}^{\mathfrak{sl}_N}(\mathcal{K}^k) \rightsquigarrow \text{KhR}^{\mathfrak{sl}_N}(\mathcal{K}^k)$$

which respects the homological and quantum grading.

3. COLORED HOMFLY-PT HOMOLOGY

In Section 3.1 we give a review of colored HOMFLY-PT homology which utilizes the notation and techniques already introduced in this paper. In Section 3.2 we transition to matrix factorization technology to study spectral sequences connecting colored HOMFLY-PT and \mathfrak{sl}_N homologies as claimed in Theorem 3. Finally, we introduce reduced colored HOMFLY-PT homology in Section 3.3 and prove Theorems 1, 2, 4 and 6.

3.1. Construction. Let \mathcal{B} be a labelled braid diagram that closes to a labelled link \mathcal{L} and consider the cube of resolutions chain complex $[\![\mathcal{B}]\!]$ in $N\mathbf{Foam}$, whose construction was sketched in Section 2.2, where we assume that N is larger than the sum of the labels on the strands of \mathcal{B} . Denote the domain and co-domain object of webs in $[\![\mathcal{B}]\!]$ by $\mathbf{o}_{\mathcal{B}} = (o_1, \dots, o_m)$. More generally, a braid, tangle or web whose left and right boundaries carry matching sequences of labels is called *balanced*.

Definition 3.1. For a web $W: \mathbf{o}_1 \rightarrow \mathbf{o}_2$ in $N\mathbf{Foam}$ we consider the sub-algebra $\text{Dec}(W)$ of the endomorphism algebra $\text{End}_{\text{foam } N}(W)$ generated by the decorated identity foams on W . For identity foams on an object $\mathbf{o} = (o_1, \dots, o_k)$ we abbreviate $\text{Dec}(\mathbf{o}) := \text{Dec}(1_{\mathbf{o}})$ and note that $\text{Dec}(W)$ is naturally a $\text{Dec}(\mathbf{o}_1) \otimes \text{Dec}(\mathbf{o}_2)$ -module by placing decorations on the right and left boundary facets.

With respect to a braid diagram \mathcal{B} as above, we define $R_{\mathcal{B}} := \text{Dec}(\mathbf{o}_{\mathcal{B}}) \cong \text{Sym}(\mathbb{X}_1 | \dots | \mathbb{X}_m)$ and remind the reader that it is given by the graded algebra of polynomials separately symmetric in o_i -element alphabets \mathbb{X}_i . Under the assignment Dec , the webs in the complex $[[\mathcal{B}]]$ correspond to $R_{\mathcal{B}}$ -bimodules, which are special cases of *singular Soergel bimodules*, see [55].

Example 3.2. The explicit description of the following decoration bimodules follows from the decoration migration relations (2.1) on foams:

$$\begin{aligned}
\text{Dec}(\text{---}\leftarrow\text{---}) &\cong \frac{\text{Sym}(\mathbb{V}|\mathbb{Y})}{\langle \mathbb{V} = \mathbb{Y} \rangle} \\
\text{Dec}(\text{---}\leftarrow\text{---}\text{---}) &\cong \left(\frac{\text{Sym}(\mathbb{V}|\mathbb{L}|\mathbb{M})}{\langle \mathbb{V} = \mathbb{L} \cup \mathbb{M} \rangle} \right) \otimes_{\text{Sym}(\mathbb{L}|\mathbb{M})} \left(\frac{\text{Sym}(\mathbb{L}|\mathbb{M}|\mathbb{Y})}{\langle \mathbb{L} \cup \mathbb{M} = \mathbb{Y} \rangle} \right) \\
\text{Dec}(\text{---}\leftarrow\text{---}\text{---}) &\cong \left(\frac{\text{Sym}(\mathbb{V}|\mathbb{W}|\mathbb{L})}{\langle \mathbb{V} \cup \mathbb{W} = \mathbb{L} \rangle} \right) \otimes_{\text{Sym}(\mathbb{L})} \left(\frac{\text{Sym}(\mathbb{L}|\mathbb{X}|\mathbb{Y})}{\langle \mathbb{L} = \mathbb{X} \cup \mathbb{Y} \rangle} \right) \\
\text{Dec}(\text{---}\leftarrow\text{---}) &\cong \frac{\text{Sym}(\mathbb{V}|\mathbb{X})}{\langle \mathbb{V} = \mathbb{X} \rangle} \otimes_{\mathbb{C}} \frac{\text{Sym}(\mathbb{W}|\mathbb{Y})}{\langle \mathbb{W} = \mathbb{Y} \rangle} \\
\text{Dec}(\text{---}\leftarrow\text{---}\text{---}) &\cong \left(\frac{\text{Sym}(\mathbb{V}|\mathbb{W}|\mathbb{L})}{\langle \mathbb{V} \cup \mathbb{W} = \mathbb{L} \rangle} \right) \otimes_{\text{Sym}(\mathbb{L})} \left(\frac{\text{Sym}(\mathbb{L}|\mathbb{X}|\mathbb{Y})}{\langle \mathbb{L} \cup \mathbb{X} = \mathbb{Y} \rangle} \right) \\
\text{Dec}(\text{---}\leftarrow\text{---}\text{---}) &\cong \left(\frac{\text{Sym}(\mathbb{W}|\mathbb{X}|\mathbb{M})}{\langle \mathbb{W} \cup \mathbb{X} = \mathbb{M} \rangle} \right) \otimes_{\text{Sym}(\mathbb{M})} \left(\frac{\text{Sym}(\mathbb{V}|\mathbb{M}|\mathbb{Y})}{\langle \mathbb{V} \cup \mathbb{M} = \mathbb{Y} \rangle} \right)
\end{aligned}$$

Here and in the following, we use the suggestive shorthand $\langle \mathbb{X}_1 \cup \dots \cup \mathbb{X}_a = \mathbb{X}_{a+1} \cup \dots \cup \mathbb{X}_{a+b} \rangle$ for ideals of the form $\langle e_i(\mathbb{X}_1 \cup \dots \cup \mathbb{X}_a) - e_i(\mathbb{X}_{a+1} \cup \dots \cup \mathbb{X}_{a+b}) | i > 0 \rangle$ which have the effect of identifying polynomials in the respective alphabets in the quotient ring.

The reader may recall the description of the bimodules in Example 3.2 from [49]. There, Rose and the author use these bimodules as a computational shortcut to relate the (deformed) link homologies constructed in foam 2-categories with the original construction using matrix factorizations [29, 60, 59, 58]. More precisely, we show that, under favourable circumstances⁹, the matrix factorizations assigned to webs in this construction are essentially determined by the bimodules $\text{Dec}(\cdot)$ via the process of stabilization. Similarly, homomorphisms of matrix factorizations are then determined by the corresponding bimodule maps. For details, see [49, Section 4.4] and the references therein, in particular [3, 5, 6, 52].

In [49, Section 4.4] we also provide a dictionary for translating foams into homomorphisms between the decoration bimodules of their domain and co-domain webs. With respect to the notation in

⁹In the absence of oriented cycles in the web.

Example 3.2 we let Dec associate bimodule homomorphisms to foams via the assignment:

$$\begin{aligned}
 & \text{Foam 1} \mapsto (\bar{\mathbb{I}} \mapsto \bar{\mathbb{I}} \otimes \bar{\mathbb{I}}) \quad , \quad \text{Foam 2} \mapsto \left(\overline{\pi_{\lambda}^{\mathbb{W}}} \otimes \bar{\mathbb{I}} \mapsto \begin{cases} \bar{\mathbb{I}} & \text{if } \lambda = b^a, \\ 0 & \text{all other } \lambda \in P(a, b), \end{cases} \right) \\
 & \text{Foam 3} \mapsto (\bar{\mathbb{I}} \mapsto \bar{\mathbb{I}} \otimes \bar{\mathbb{I}}) \quad , \quad \text{Foam 4} \mapsto \left(\bar{\mathbb{I}} \otimes \bar{\mathbb{I}} \mapsto \sum_{\alpha \in P(a, b)} (-1)^{|\hat{\alpha}|} \overline{\pi_{\alpha}^{\mathbb{W}}} \otimes \pi_{\alpha}^{\mathbb{Y}} \right) \\
 & \text{Foam 5} \mapsto (\bar{\mathbb{I}} \otimes \bar{\mathbb{I}} \mapsto \bar{\mathbb{I}} \otimes \bar{\mathbb{I}}) \quad , \quad \text{Foam 6} \mapsto (\bar{\mathbb{I}} \otimes \bar{\mathbb{I}} \mapsto \bar{\mathbb{I}} \otimes \bar{\mathbb{I}})
 \end{aligned}$$

Here \bar{f} denotes an element of a quotient ring represented by a polynomial f , $\pi_{\lambda}^{\mathbb{W}}$ denotes the Schur polynomial in the alphabet \mathbb{W} corresponding to the Young diagram λ , $\hat{\lambda}$ denotes the dual of its complementary diagram, $P(a, b)$ is the set of Young diagrams fitting into a box of size $a \times b$ and its maximal element is b^a .

Proposition 3.3. *The above assignments uniquely define bimodule maps between the corresponding decoration bimodules. Dec extends naturally to a monoidal 2-functor from $N\mathbf{Foam}$ to the 2-category \mathbf{Bimod} of singular Soergel bimodules. For a balanced labelled braid diagram \mathcal{B} , the complex $\text{Dec}(\llbracket \mathcal{B} \rrbracket)$ is isomorphic to the complex of singular Soergel bimodules constructed by Mackaay-Stošić-Vaz [36] and Webster-Williamson [53].*

Proof. Uniqueness is clear since the maps are defined on a generating set and the bimodule map condition can be checked explicitly. All arising decoration bimodules are singular Soergel bimodules which form a 2-category \mathbf{Bimod} and [49, Section 4.4] contains a proof that the images of foam relations under Dec hold in \mathbf{Bimod} . Moreover, Dec respects the monoidal structure given by disjoint union on foams and by tensor product over \mathbb{C} on bimodules. A quick comparison with [36] shows that the complex constructed there and $\text{Dec}(\llbracket \mathcal{B} \rrbracket)$ have the same bimodules as objects. Furthermore, in both cases, the differentials are constructed locally for each crossing and it is well-known that the differentials in a crossing complex of singular Soergel bimodules are uniquely determined (up to non-zero scalars), see e.g. [53, Corollary 5.7.]. Thus, the complexes are isomorphic. \square

Next, one needs a functor to translate $\text{Dec}(\llbracket \mathcal{B} \rrbracket)$ into a complex of triply-graded vector spaces.

Definition 3.4. Let \mathcal{R} be a commutative, graded \mathbb{C} -algebra. The Hochschild homology of a graded \mathcal{R} -bimodule M (equivalently, a $\mathcal{R} \otimes \mathcal{R}$ -module) is defined to be:

$$HH_*(M) := \text{Tor}_*^{\mathcal{R} \otimes \mathcal{R}}(\mathcal{R}, M)$$

In fact, $HH_*(\cdot)$ is covariant functor from the category of graded \mathcal{R} -bimodules to the category of bi-graded \mathbb{C} -vector spaces.

The following equation shows the complex computing the corresponding Hochschild homology, where we have indicated grading shifts by powers of q and the slightly unnaturally looking variable $-a^2$. It is clear from the above description that the differentials are all zero.

$$\mathrm{HH}_*(\mathrm{Dec}(\overleftarrow{\text{---}2})) = a^{-4} \mathrm{H}_* \left(\begin{array}{ccc} q^2 \mathrm{Dec}(\overleftarrow{\text{---}2}) & \xrightarrow{0} & -a^2 q^2 \mathrm{Dec}(\overleftarrow{\text{---}2}) \\ & \searrow 0 & \nearrow 0 \\ & -a^2 \mathrm{Dec}(\overleftarrow{\text{---}2}) & \xrightarrow{0} a^4 \mathrm{Dec}(\overleftarrow{\text{---}2}) \end{array} \right)$$

The Hochschild homology of $\mathrm{Dec}(\overleftarrow{\text{---}2})$ thus consists of four grading shifted copies of $\mathrm{Dec}(\overleftarrow{\text{---}2})$ itself, whose underlying \mathbb{C} -space is isomorphic to a polynomial algebra with generators of internal degree 2 and 4. Thus, the corresponding Hilbert–Poincaré series has the expansion

$$\mathrm{HP}(\mathrm{HH}_*(\mathrm{Dec}(\overleftarrow{\text{---}2}))) = a^{-4} \frac{(a^4 - a^2 q^2 - a^2 + q^2)}{(1 - q^2)(1 - q^4)} = a^{-4} \frac{(a^2 - 1)(a^2 - q^2)}{(1 - q^2)(1 - q^4)}$$

Up to overall multiplication by a monomial, this agrees with the HOMFLY-PT polynomial of the 2-labelled unknot.

3.2. HOMFLY-PT to \mathfrak{sl}_N spectral sequences. Following Webster [52], we explain how the Hochschild homology functor can be replaced by an alternative functor in order to compute the Σ -deformed Khovanov–Rozansky homology instead of HOMFLY-PT homology from $\mathrm{Dec}[\mathcal{B}]$. This functor takes the form

$$(3.3) \quad WH_*^\Sigma(\cdot) := H_{tot}(\cdot \otimes \mathcal{Z}_{\mathbf{y}, \mathbf{x}}).$$

see Theorem 3.20. Here, the Koszul complex $\mathcal{Z}_{\mathbf{x}}$ has been replaced by a *Koszul matrix factorization* $\mathcal{Z}_{\mathbf{y}, \mathbf{x}}$. We now recall some background on matrix factorizations and their relevance for \mathfrak{sl}_N link homologies.

Definition 3.10. Let \mathcal{R} be a commutative \mathbb{C} -algebra. A \mathbb{Z} -graded matrix factorization over \mathcal{R} with potential $\varphi \in \mathcal{R}$ is a \mathbb{Z} -graded \mathcal{R} -module M together with a map $d_{tot} = d_+ + d_- : M \rightarrow M$ composed of differentials d_\pm of degree ± 1 such that d_{tot}^2 acts by multiplication by φ . A morphism of matrix factorizations is a graded \mathcal{R} -module map which intertwines the respective d_+ ’s and d_- ’s. Moreover, matrix factorizations over \mathcal{R} admit a tensor product which is additive on potentials. We call the \mathbb{Z} -grading the *horizontal* grading and denote it by gr_h . Usually, the matrix factorizations we consider will have an additional internal polynomial grading denoted gr_q .

We use the shorthand notation $M^+ = (M, d_+)$ for M considered as a chain complex with differential d_+ . A matrix factorization with potential zero is also a chain complex with respect to $d_{tot} = d_+ + d_-$, which implies that d_+ and d_- anti-commute. In this case, we write $H_{tot}(M) := H_*(M, d_{tot})$ for the total homology, but we could also consider just the positive homology with its induced d_-^* differential $(H^+(M), d_-^*) := (H_*(M^+), d_-^*)$.

Definition 3.11. Let $\mathbf{a} = \{a_1, \dots, a_n\}$ and $\mathbf{b} = \{b_1, \dots, b_n\}$ be sequences of elements of \mathcal{R} , then the *Koszul matrix factorization* of \mathbf{a} and \mathbf{b} is defined as:

$$\mathcal{Z}_{\mathbf{a}, \mathbf{b}} := \bigotimes_{i=1}^n \left(\mathcal{R} \begin{array}{c} \xrightarrow{b_i} \\ \xleftarrow{a_i} \end{array} \mathcal{R} \right)$$

This is a matrix factorization with potential $\sum_{i=1}^n a_i b_i$. If \mathcal{R} is graded and the b_i homogeneous, we always assume that degrees are shifted to make d_+ homogeneous of internal degree 2. Note that then $\mathcal{Z}_{\mathbf{a}, \mathbf{b}}^+ = \mathcal{Z}_{\mathbf{b}}$. It is common to encode the differentials of $\mathcal{Z}_{\mathbf{a}, \mathbf{b}}$ in a $n \times 2$ matrix with columns \mathbf{a} and \mathbf{b} . Then the rows of this matrix correspond to the tensor factors in the defining equation.

Lemma 3.12. [47, Section 3.3, row operations] *Let $a_1, a_2, b_1, b_2, c \in \mathcal{R}$. Then there is an isomorphism of \mathbb{Z} -graded matrix factorizations: $\mathcal{Z}_{a_1, b_1} \otimes \mathcal{Z}_{a_2, b_2} \cong \mathcal{Z}_{a_1 + ca_2, b_1} \otimes \mathcal{Z}_{a_2, b_2 - cb_1}$.*

More generally, we can obtain isomorphisms of larger Koszul matrix factorizations by applying a sequence of such row operations on pairs of tensor factors in them. Recall that a sequence of elements $\mathbf{b} = \{b_1, \dots, b_k\}$ of \mathcal{R} is *regular* if for $1 \leq i \leq k$ the element b_i is not a zero-divisor in $\mathcal{R}/\langle b_1, \dots, b_{i-1} \rangle$. It is easy to see that $H^+(\mathcal{Z}_{\mathbf{b}}) \cong \mathcal{R}/\langle \mathbf{b} \rangle$, concentrated in horizontal degree zero. A generalization of this fact is provided by the following result of Webster.

Proposition 3.13. [52, Section 1.2] *Let $\mathcal{Z}_{\mathbf{a}, \mathbf{b}}$ be a Koszul matrix factorization over \mathcal{R} with potential φ . Suppose that \mathbf{b} is a regular sequence in \mathcal{R} . Let M be a projective matrix factorization over \mathcal{R} with potential $-\varphi$. Then the chain map $\pi: \mathcal{Z}_{\mathbf{a}, \mathbf{b}}^+ \rightarrow H^+(\mathcal{Z}_{\mathbf{a}, \mathbf{b}}) \cong \mathcal{R}/\langle \mathbf{b} \rangle$ induces the following isomorphisms of \mathcal{R} -modules:*

$$\begin{aligned} H^+(\mathcal{Z}_{\mathbf{a}, \mathbf{b}} \otimes M) &\cong H^+(\mathcal{R}/\langle \mathbf{b} \rangle \otimes M) \\ H_{tot}(\mathcal{Z}_{\mathbf{a}, \mathbf{b}} \otimes M) &\cong H_{tot}(\mathcal{R}/\langle \mathbf{b} \rangle \otimes M) \end{aligned}$$

Corollary 3.14. *Under the assumptions of Proposition 3.13, there is an isomorphism of chain complexes of \mathcal{R} -modules:*

$$(H^+(\mathcal{Z}_{\mathbf{a}, \mathbf{b}} \otimes M), d_-^*) \cong (H^+(\mathcal{R}/\langle \mathbf{b} \rangle \otimes M), d_-^*)$$

Proof. The chain map $\pi: \mathcal{Z}_{\mathbf{a}, \mathbf{b}}^+ \rightarrow H^+(\mathcal{Z}_{\mathbf{a}, \mathbf{b}}) \cong \mathcal{R}/\langle \mathbf{b} \rangle$ also intertwines the negative differential d_- on $\mathcal{Z}_{\mathbf{a}, \mathbf{b}}$ with the trivial negative differential on $\mathcal{R}/\langle \mathbf{b} \rangle$, i.e. it is a morphism of matrix factorizations. Then $\pi \otimes 1: \mathcal{Z}_{\mathbf{a}, \mathbf{b}} \otimes M \rightarrow \mathcal{R}/\langle \mathbf{b} \rangle \otimes M$ also intertwines the negative differentials. The induced map on positive homology is an isomorphism on modules which intertwines the induced differentials d_-^* , i.e. it is an isomorphism of chain complexes. \square

Definition 3.15. Let $\mathcal{Z}_{\mathbf{a}, \mathbf{b}}$ be as in Proposition 3.13 and M_1 a projective matrix factorization over \mathcal{R} (not necessarily with potential $-\varphi$), then we consider $\mathcal{R}/\langle \mathbf{b} \rangle \otimes M_1$ as a matrix factorization over $\mathcal{R}/\langle \mathbf{b} \rangle$ and call it a *simplification* of $\mathcal{Z}_{\mathbf{a}, \mathbf{b}} \otimes M_1$.

Proposition 3.13 and Corollary 3.14 imply that Koszul matrix factorizations appearing as tensor factors in a (projective) matrix factorization with potential zero can be replaced by their simplifications without changing the H^+ homology with its induced negative differential or the H_{tot} homology or the \mathcal{R} -module structures on them.

We now recall how to assign Koszul matrix factorizations to webs, which can be used to compute \mathfrak{sl}_N link homologies, see [59, 58]¹⁰. For this, let W be a web and associate to every edge in W with label l an alphabet \mathbb{Y} with $|\mathbb{Y}| = l$. The alphabets adjacent to the left and right boundary we again denote by \mathbb{X}_i and \mathbb{X}'_i . We consider the ring \mathcal{R}_W of polynomials separately symmetric in these alphabets. For a polynomial $T(X) = \sum_{i=0}^k c_i X^i \in \mathbb{C}[X]$ we set $T(\mathbb{X}) = \sum_{i=0}^k c_i p_i(\mathbb{X})$ where $p_i(\mathbb{X})$ denotes the i^{th} power sum symmetric polynomial in the alphabet \mathbb{X} . Recall that we denote by $P(X) \in \mathbb{C}[X]$ the monic polynomial with root multiset Σ and let $Q(X) \in \mathbb{C}[X]$ be such that $Q'(X) = (N+1)P(X)$ and $Q(0) = 0$.

Definition 3.16. To every web W we associate a Koszul matrix factorization $\text{MF}^\Sigma(W)$ over \mathcal{R}_W with potential $Q(\bigsqcup \mathbb{X}'_j) - Q(\bigsqcup \mathbb{X}_j)$. We first describe it in special cases:

- If W is a single strand of label a without incident vertices, then $\text{MF}^\Sigma(W)$ has positive sequence $e_j(\mathbb{X}'_1) - e_j(\mathbb{X}_1)$ for $1 \leq j \leq a$ and the negative sequence is chosen¹¹ to generate the potential $Q(\mathbb{X}'_1) - Q(\mathbb{X}_1)$.
- If W is a merge web of incoming strands of thickness a and b then $\text{MF}^\Sigma(W)$ has the positive sequence consisting of the entries $e_j(\mathbb{X}'_1) - e_j(\mathbb{X}_1 \sqcup \mathbb{X}_2)$ for $1 \leq j \leq a+b$ and the negative sequence is chosen to generate the potential $Q(\mathbb{X}'_1) - Q(\mathbb{X}_1 \sqcup \mathbb{X}_2)$.

¹⁰Note however that we use \mathbb{Z} -graded matrix factorizations instead of \mathbb{Z}_2 -graded ones.

¹¹It is easy to see that this is possible, the specific choice is irrelevant for our discussion.

- Analogously, the Koszul matrix factorization associated to the split web has the positive sequence with entries $e_j(\mathbb{X}'_1 \sqcup \mathbb{X}'_2) - e_j(\mathbb{X}_1)$ and an appropriate negative sequence.

For any other web W , the Koszul matrix factorization $\mathrm{MF}^\Sigma(W)$ is defined over \mathcal{R}_W and consists of all the rows of appearing in the Koszul factorizations associated to the merge and split vertices and non-interacting strands in W .

Lemma 3.17. *Let W be a web in $[\![\mathcal{B}]\!]$. Then $\mathrm{MF}^\Sigma(W)$ simplifies to $H^+(\mathrm{MF}^\Sigma(W)) \cong \mathrm{Dec}(W)$ and this simplification is concentrated in horizontal degree zero.*

Proof. It is easy to check that the positive sequence of $\mathrm{MF}^\Sigma(W)$ is regular, and so the positive homology is concentrated in degree zero and isomorphic to the quotient of the ground ring by the ideal generated by the elements in the sequence. Identifying the ring \mathcal{R}_W with the possible placements of decorations on the identity foam on W , the entries of the positive sequence generate the same ideal as relations (2.1). This identifies $H^+(\mathrm{MF}^\Sigma(W))$ with $\mathrm{Dec}(W)$. \square

Remark 3.18. Lemma 3.17 shows that we can view $\mathrm{Dec}(W)$ as a matrix factorization over $\mathcal{R}_\mathcal{B} \otimes \mathcal{R}_\mathcal{B}$ with potential $\varphi := Q(\bigsqcup \mathbb{X}_j) - Q(\bigsqcup \mathbb{X}'_j)$ and trivial differentials. To see this directly, note that since $H^+(\mathrm{MF}^\Sigma(W))$ is concentrated in horizontal degree zero, we have $\mathrm{Dec}(W) \cong \mathrm{MF}_0^\Sigma(\mathcal{B})/\mathrm{Im}d_+$. Then $\varphi \mathrm{id}_{\mathrm{MF}_0^\Sigma(\mathcal{B})} = d_+d_-$ and so the element φ is in the image of d_+ and acts by zero.

In [49, Section 4.4] Rose and the author show that there exists a 2-functor from $N\mathbf{Foam}^\Sigma$ to a 2-category of $(\mathbb{Z}_2\text{-graded})$ matrix factorizations with potential modelled on Q . Under this 2-functor, the web W is sent to the matrix factorization $\mathrm{MF}^\Sigma(W)$ (with the horizontal grading taken modulo 2). More generally, for a braid \mathcal{B} , the complex $[\![\mathcal{B}]\!]^\Sigma$ is sent to a *cube of resolutions* complex $\mathrm{MF}^\Sigma(\mathcal{B})$ of matrix factorizations and morphisms between them. This complex agrees up to isomorphism and grading conventions with the complex used in Wu's construction of (deformed) colored \mathfrak{sl}_N link homologies in [59, 58]. We will not describe the differentials in this complex more explicitly, as Lemma 3.17 implies that it suffices to work with $\mathrm{Dec}([\![\mathcal{B}]\!])$, see Theorem 3.20.

We fix the notation $\mathcal{Z}_{\mathbf{y},\mathbf{x}} := \mathrm{MF}^\Sigma(1_{\mathbf{o}_\mathcal{B}}^{\mathrm{op}})$ for the Koszul matrix factorization assigned to the oppositely oriented identity web on $\mathbf{o}_\mathcal{B}$, i.e. with alphabets \mathbb{X}_j at the left and alphabets \mathbb{X}'_j at the right boundary. It is defined over $\mathcal{R}_\mathcal{B} \otimes \mathcal{R}_\mathcal{B}$, has potential $Q(\bigsqcup_j \mathbb{X}_j) - Q(\bigsqcup_j \mathbb{X}'_j)$ and the positive sequence \mathbf{x} consists of entries $e_l(\mathbb{X}_j) - e_l(\mathbb{X}'_j)$ for $1 \leq j \leq m$ and $1 \leq l \leq o_j$.

Definition 3.19. The matrix factorization associated to the labelled link diagram \mathcal{L} given as closure of a balanced labelled braid diagram \mathcal{B} is a complex of Koszul matrix factorizations with potential zero defined as

$$\mathrm{MF}^\Sigma(\mathcal{L}) := \mathrm{MF}^\Sigma(\mathcal{B}) \otimes \mathcal{Z}_{\mathbf{y},\mathbf{x}}.$$

Further, we define the functor

$$WH_*^\Sigma(\cdot) := H_{\mathrm{tot}}(\cdot \otimes \mathcal{Z}_{\mathbf{y},\mathbf{x}})$$

from the category of matrix factorizations over $\mathcal{R}_\mathcal{B} \otimes \mathcal{R}_\mathcal{B}$ with potential $Q(\bigsqcup_j \mathbb{X}'_j) - Q(\bigsqcup_j \mathbb{X}_j)$ to the category of (bi-)graded $\mathcal{R}_\mathcal{B}$ -modules.

The functor $WH_*^\Sigma(\cdot)$ is a matrix factorization analogue of the Hochschild homology functor $HH_*(\cdot)$ which depends on the multiset Σ , see Lemma 3.6. It has been introduced by Webster in [52]. Clearly, $WH_*^\Sigma(\cdot)$ extends to chain complexes of matrix factorizations, in which case we denote the induced *vertical differential* by d_v^* and the new homological, a.k.a. *vertical grading* by gr_v .

Theorem 3.20. *Let \mathcal{B} be a balanced labelled braid diagram and \mathcal{L} its closure. Then we have:*

$$(3.4) \quad \mathrm{KhR}^\Sigma(\mathcal{L}) \cong H_*(H_{\mathrm{tot}}(\mathrm{MF}^\Sigma(\mathcal{L})), d_v^*) \cong H_*(WH_*^\Sigma(\mathrm{Dec}([\![\mathcal{B}]\!])), d_v^*)$$

Proof. First of all, we note that [49, Theorem 2] identifies $\mathrm{KhR}^\Sigma(\mathcal{L})$ with Wu's Σ -deformed Khovanov Rozansky \mathfrak{sl}_N homology, which can be computed as

$$\mathrm{KhR}^\Sigma(\mathcal{L}) \cong H_*(H_{tot}(\mathrm{MF}^\Sigma(\mathcal{L})), d_v^*)$$

where the inner homology is with respect to the total matrix factorization differential $d_{tot} = d_+ + d_-$ and the outer homology is with respect to the induced vertical differential d_v^* . This establishes the first isomorphism. For the second isomorphism we use that the vertical complex $\mathrm{MF}^\Sigma(\mathcal{B})$ simplifies to $H^+(\mathrm{MF}^\Sigma(\mathcal{B})) \cong \mathrm{Dec}(\llbracket \mathcal{B} \rrbracket)$.

$$H_{tot}(\mathrm{MF}^\Sigma(\mathcal{L})) = H_{tot}(\mathrm{MF}^\Sigma(\mathcal{B}) \otimes \mathcal{Z}_{\mathbf{y}, \mathbf{x}}) \cong H_{tot}(\mathrm{Dec}(\llbracket \mathcal{B} \rrbracket) \otimes \mathcal{Z}_{\mathbf{y}, \mathbf{x}}) = WH_*^\Sigma(\mathrm{Dec}(\llbracket \mathcal{B} \rrbracket))$$

and the second claimed isomorphism follows after taking d_v^* -homology. \square

Remark 3.21. We need to comment on the behaviour of gradings under the identifications in (3.4). Up to a global shift, the homological grading of $\mathrm{KhR}^\Sigma(\mathcal{L})$ simply agrees with the homological grading on $H_*(WH_*^\Sigma(\mathrm{Dec}(\llbracket \mathcal{B} \rrbracket)), d_v^*)$. In the undeformed case where $\Sigma = \{0, \dots, 0\}$, we also would like to recover the quantum grading of $\mathrm{KhR}^\Sigma(\mathcal{L}) \cong \mathrm{KhR}^{\mathfrak{sl}_N}(\mathcal{L})$ on the right-hand side of (3.4). Note however, that $d_{tot} = d_+ + d_-$ is not even homogeneous with respect to the internal polynomial grading gr_q on $H_*(WH_*^\Sigma(\mathrm{Dec}(\llbracket \mathcal{B} \rrbracket)), d_v^*)$. Instead one considers the new grading $gr_N := gr_q + (N-1)gr_h$, which is preserved by d_{tot} and thus descends to homology, where it can be shown to correspond (up to a global shift) to the quantum grading on the left-hand side of (3.4), see Rasmussen's [47, Proposition 3.12].

Theorem 3.22. *There is a spectral sequence*

$$\mathrm{KhR}^\infty(\mathcal{L}) \rightsquigarrow \mathrm{KhR}^\Sigma(\mathcal{L})$$

whose k^{th} differential lowers the horizontal and homological grading by k and $k-1$ respectively. If $\Sigma = \{0, \dots, 0\}$, it increases the internal degree by $2kN$.

Sketch proof following Rasmussen [47]. The matrix factorization $\mathrm{MF}^\Sigma(\mathcal{L})$ associated to \mathcal{L} is a complex equipped with the three anti-commuting differentials d_+ , d_- and d_v . Its H^+ -homology is a triply-graded chain complex (internal grading, horizontal grading, homological grading), with induced differentials d_+^* and d_v^* , which lower horizontal and homological grading by one respectively, but preserve the other grading. The horizontal filtration on this double complex induces a spectral sequence with E_0 page $(H^+(\mathrm{MF}^\Sigma(\mathcal{L})), d_v^*)$, which is isomorphic to $(HH_*(\mathrm{Dec}(\llbracket \mathcal{B} \rrbracket)), d_v^*)$ by Proposition 3.13. Consequently, the E_1 page is isomorphic to $H_*(HH_*(\mathrm{Dec}(\llbracket \mathcal{B} \rrbracket)), d_v^*) = \mathrm{KhR}^\infty(\mathcal{L})$. The spectral sequence converges to the total homology $H_*((H^+(\mathrm{MF}^\Sigma(\mathcal{L})), d_+^* + d_v^*))$ and it remains to identify this with $\mathrm{KhR}^\Sigma(\mathcal{L}) = H_*(H_{tot}(\mathrm{MF}^\Sigma(\mathcal{L})), d_v^*)$. The proof of this is split into two steps. First, we need

$$H_*((H^+(\mathrm{MF}^\Sigma(\mathcal{L})), d_+^* + d_v^*)) \cong H_*(H_*((H^+(\mathrm{MF}^\Sigma(\mathcal{L})), d_+^*), d_v^*),$$

which can be proved exactly as in [47, Proposition 5.10]. And second

$$H_*(H_*((H^+(\mathrm{MF}^\Sigma(\mathcal{L})), d_+^*), d_v^*)) \cong H_*(H_{tot}(\mathrm{MF}^\Sigma(\mathcal{L})), d_v^*),$$

which follows as in [47, Lemma 5.11, Lemma 5.12, Proposition 5.13]. The proofs of these isomorphisms, however, use a critical fact, which is a generalization of Rasmussen's [47, Proposition 5.8 and Corollary 5.9] and which we prove in the separate Lemma 3.23.

As regards gradings, the k^{th} differential in the spectral sequence lowers horizontal degree gr_h by k and homological degree gr_v by $k-1$ (and it raises internal degree gr_q by $2kN$ in the undeformed case). In particular, all differentials in the spectral sequence preserve the quantity $gr_- = gr_v - gr_h$ (and $gr'_N = gr_q + 2Ngr_h$), inducing a grading on the E_∞ page. Careful comparison of gradings as in [47, Proposition 5.14] shows that this agrees with homological grading (and quantum grading) on $\mathrm{KhR}^\Sigma(\mathcal{K})$ (and $\mathrm{KhR}^{\mathfrak{sl}_N}(\mathcal{K})$ respectively) as identified in Remark 3.21. \square

Lemma 3.23. $H_*(H^+(\mathrm{MF}^\Sigma(\mathcal{L})), d_-^*)$ is concentrated in a single horizontal grading.

Proof. An analogue of Rasmussen's [47, Corollary 5.9] shows that it suffices to prove the claim for a matrix factorization $\mathrm{MF}^\Sigma(L)$ in place of $\mathrm{MF}^\Sigma(\mathcal{L})$, where L is the annular closure of a web W appearing in $\llbracket \mathcal{B} \rrbracket$. The key ingredient in Rasmussen's proof of this fact in the uncolored setting in [47, Proposition 5.8] is a recursive algorithm for the computation of $H_*(H^+(\mathrm{MF}^\Sigma(L)), d_-^*)$, see [47, Sections 4.2 and 4.3]. A suitable replacement for this algorithm in the colored case is an analogue of the annular web evaluation algorithm of Queffelec and Rose [43, Proposition 5.1 and Lemma 5.2], which we review in detail in Section 4. In its simplest, decategorified version, the algorithm expands an annular web as a linear combination of nested labelled essential circles. It does so by successively rewriting annular webs as linear combinations of annular webs of lower complexity using one of four types of relations: sliding ladder rungs around the annulus, ladder rung combination, Frobenius relations and square switches.

The categorified version of this algorithm operates in an annular foam category and shows that the annular closure of the web W appears as a direct summand of an object isomorphic to a direct sum of nested labelled essential circles. The first three types of relations have direct counterparts in the annular foam category, which allow a web to be replaced by an isomorphic direct sum of webs of lower complexity. The square switch relation [43, (5.7)] is more subtle, as it only allows to replace a web by a direct summand of an object isomorphic to a direct sum of webs of lower complexity¹².

By following this algorithm, we would like to show that $H_*(H^+(\mathrm{MF}^\Sigma(L)), d_-^*)$ lies in one horizontal grading. So we need that the above four relations between annular webs also hold after applying $H_*(H^+(\mathrm{MF}^\Sigma(\cdot), d_-^*))$. For the rung sliding relation, this follows directly from the definition of $\mathrm{MF}^\Sigma(\cdot)$ and Proposition 3.13. For the other three relations we write

$$(H^+(\mathrm{MF}^\Sigma(L)), d_-^*) = (H^+(\mathrm{MF}^\Sigma(W) \otimes \mathcal{Z}_{\mathbf{y}, \mathbf{x}}), d_-^*) \stackrel{\text{Cor. 3.14}}{\cong} (H^+(\mathrm{Dec}(W) \otimes \mathcal{Z}_{\mathbf{y}, \mathbf{x}}), d_-^*)$$

and see that rung combination, the Frobenius relations and square switch relations hold here because they hold for singular Soergel bimodules $\mathrm{Dec}(\cdot)$, as verified e.g. in [49, Section 4.4]. More precisely, an isomorphism of Soergel bimodules (i.e. of $\mathcal{R}_{\mathcal{B}} \otimes \mathcal{R}_{\mathcal{B}}$ -modules) induces an isomorphism of matrix factorizations after tensoring with $\mathcal{Z}_{\mathbf{y}, \mathbf{x}}$ over $\mathcal{R}_{\mathcal{B}} \otimes \mathcal{R}_{\mathcal{B}}$. After taking positive homology, we get an isomorphism of complexes with respect to the induced negative differentials.

We conclude that $H_*(H^+(\mathrm{MF}^\Sigma(L)), d_-^*)$ is (isomorphic to) a sub-module of a direct sum of objects of the form $H_*(H^+(\mathrm{MF}^\Sigma(C)), d_-^*)$, where C is a collection of nested labelled essential circles. Such a summand is concentrated in the horizontal grading equal to the negative of the sum of the labels on the circles in C , which is equal (to $\sum o_i$) for all C appearing in the annular evaluation algorithm. Thus $H_*(H^+(\mathrm{MF}^\Sigma(L)), d_-^*)$ and then also $H_*(H^+(\mathrm{MF}^\Sigma(\mathcal{L})), d_-^*)$ are concentrated in this grading as well. \square

Remark 3.24. The categorified annular web evaluation algorithm of Queffelec-Rose [43], which is essentially independent of N , suggests a possible alternative approach to colored HOMFLY-PT homology as an invariant of annular links. This might lead to a new and more illuminating proof of Theorem 3.22.

3.3. Reduced colored HOMFLY-PT homology. In this section, we introduce a reduced version of colored HOMFLY-PT homology and relate it to reduced colored \mathfrak{sl}_N homology. As in the case of \mathfrak{sl}_N homology, we have to choose a link component at which we want to reduce. To this end, we consider a balanced labelled braid diagram \mathcal{B} – which closes to an oriented, labelled link diagram \mathcal{L} – together with a marked right boundary point with associated alphabet \mathbb{X}_i of size $k = |\mathbb{X}_i|$. In the following, we assume that all strands of \mathcal{B} have labels $\geq k$ and that \mathcal{B} does not split into a tensor product of two braids.

¹²This issue already appears in Rasmussen's MOY III relation [47, Proposition 4.6]

Remark 3.25. Both of these assumptions are automatically satisfied for braid diagrams representing knots. Unfortunately, however, they currently prevent the treatment of colored HOMFLY-PT homologies of links reduced at a component of non-minimal label, for which interesting *color-shifting properties* have been conjectured in [20].

Definition 3.26. We cut the labelled planar graph underlying the diagram \mathcal{B} into labelled arcs between crossings, and we assign alphabets $\mathbb{X}'_1, \dots, \mathbb{X}'_m$ and $\mathbb{X}_1, \dots, \mathbb{X}_m$ to boundary arcs on the left and the right respectively and alphabets \mathbb{Y}_l of the appropriate size to the internal arcs. Note that every such arc corresponds to an edge in any web appearing in $[\mathcal{B}]$. Let

$$\mathcal{S}_{\mathcal{B}} := \text{Sym}(\mathbb{X}_1 | \dots | \mathbb{X}_m | \mathbb{X}'_1 | \dots | \mathbb{X}'_m | \mathbb{Y}_1 | \dots)$$

be the graded ring of polynomials separately symmetric in these variables. Let

$$\mathcal{V}_{\mathcal{B}} := \mathcal{S}_{\mathcal{B}} / \langle e_j(\mathbb{W}'_1 \sqcup \mathbb{W}'_2) - e_j(\mathbb{W}_1 \sqcup \mathbb{W}_2) | 1 \leq j \leq k \rangle$$

where $(\mathbb{W}'_1, \mathbb{W}'_2, \mathbb{W}_1, \mathbb{W}_2)$ ranges over all tuples of alphabets adjacent to a crossing site, with the dashed alphabets associated to outgoing edges and the other ones to incoming edges. Finally, we define

$$\mathcal{T}_{\mathcal{B}} := \mathcal{R}_{\mathcal{B}} \otimes \mathcal{R}_{\mathcal{B}} / \langle e_j(\bigsqcup_{l=1}^m \mathbb{X}'_l) - e_j(\bigsqcup_{l=1}^m \mathbb{X}_l) | 1 \leq j \leq k \rangle$$

where we identify $\mathcal{R}_{\mathcal{B}} \otimes \mathcal{R}_{\mathcal{B}} \cong \text{Sym}(\mathbb{X}_1 | \dots | \mathbb{X}_m | \mathbb{X}'_1 | \dots | \mathbb{X}'_m)$.

Lemma 3.27. $\mathcal{V}_{\mathcal{B}}$ is a $\mathcal{T}_{\mathcal{B}}$ -module.

Proof. $\mathcal{V}_{\mathcal{B}}$ is a $\mathcal{R}_{\mathcal{B}} \otimes \mathcal{R}_{\mathcal{B}}$ -module and it is easy to check (e.g. by induction on the number of crossings in \mathcal{B}) that the relations $e_j(\bigsqcup_{l=1}^m \mathbb{X}'_l) = e_j(\bigsqcup_{l=1}^m \mathbb{X}_l)$ hold in $\mathcal{V}_{\mathcal{B}}$ for $1 \leq j \leq k$. \square

Lemma 3.28. $\mathcal{T}_{\mathcal{B}}$ and $\mathcal{V}_{\mathcal{B}}$ are isomorphic to polynomial rings over \mathbb{C} and $\mathcal{V}_{\mathcal{B}}$ is free over $\mathcal{T}_{\mathcal{B}}$.

Proof. The unital \mathbb{C} -algebra $\mathcal{T}_{\mathcal{B}}$ has a presentation with commuting generators $e_j(\mathbb{X}_l)$ and $e_j(\mathbb{X}'_l)$ for $1 \leq l \leq m$ and $1 \leq j \leq o_l = |\mathbb{X}_l|$ and one relation for each $1 \leq j \leq k$, which we rewrite as follows:

$$0 = e_j(\bigsqcup_{l=1}^m \mathbb{X}'_l) - e_j(\bigsqcup_{l=1}^m \mathbb{X}_l) = e_j(\mathbb{X}'_i) - \left(e_j(\bigsqcup_{l=1}^m \mathbb{X}_l) - \sum_{x=0}^{j-1} e_x(\mathbb{X}'_i) e_{j-x}(\bigsqcup_{l \neq i} \mathbb{X}'_l) \right)$$

Thus, the generator $e_j(\mathbb{X}'_i)$ can be expressed in terms of other generators. Under this substitution, we obtain a presentation for $\mathcal{T}_{\mathcal{B}}$ with one generator and one relation eliminated. By iteration, we arrive at a presentation of $\mathcal{T}_{\mathcal{B}}$ without any relations and with all generators $e_j(\mathbb{X}'_i)$ for $1 \leq j \leq k$ eliminated. In other words, the ring homomorphism, which is given by the composition

$$(3.5) \quad \text{Sym}(\mathbb{X}_1 | \dots | \mathbb{X}_m | \mathbb{X}'_1 | \dots | \mathbb{X}'_{i-1} | \mathbb{X}'_{i+1} | \dots | \mathbb{X}'_m) \hookrightarrow \mathcal{R}_{\mathcal{B}} \otimes \mathcal{R}_{\mathcal{B}} \rightarrow \mathcal{T}_{\mathcal{B}}$$

of the canonical inclusion and the defining quotient map, is an isomorphism.

In Definition 3.26, $\mathcal{V}_{\mathcal{B}}$ is given by a similar presentation with generators associated to arcs in the braid diagram \mathcal{B} and k relations for every crossing in \mathcal{B} . For a crossing with adjacent alphabets $(\mathbb{W}'_1, \mathbb{W}'_2, \mathbb{W}_1, \mathbb{W}_2)$, we can choose one of these alphabets, call it \mathbb{W} , and eliminate the generators $e_j(\mathbb{W})$ for $1 \leq j \leq k$ in pairs with the relations $e_j(\mathbb{W}'_1 \sqcup \mathbb{W}'_2) - e_j(\mathbb{W}_1 \sqcup \mathbb{W}_2)$ for $1 \leq j \leq k$. This is possible since we assume that $|\mathbb{W}| \geq k$. To eliminate the relations associated to all crossings, we need to choose different alphabets \mathbb{W} at every crossing. It is easy to see that this is possible. In fact, since \mathcal{B} is a non-split braid diagram, we can arrange to eliminate generators $e_j(\mathbb{X}'_i)$ for $1 \leq j \leq k$ first and only generators in internal alphabets \mathbb{Y}_l thereafter. The surviving generators in internal alphabets generate a (possibly trivial) polynomial ring R and the canonical composition

$$(3.6) \quad \text{Sym}(\mathbb{X}_1 | \dots | \mathbb{X}_m | \mathbb{X}'_1 | \dots | \mathbb{X}'_{i-1} | \mathbb{X}'_{i+1} | \dots | \mathbb{X}'_m) \otimes_{\mathbb{C}} R \hookrightarrow \mathcal{S}_{\mathcal{B}} \rightarrow \mathcal{V}_{\mathcal{B}}$$

For the following we write $\mathcal{Z}_{\mathbf{y}, \mathbf{x}} \cong \mathcal{Z}_{\mathbf{y}_i, \mathbf{x}_i} \otimes_{\mathbb{Q}} \mathcal{Z}_{\bar{\mathbf{y}}, \bar{\mathbf{x}}}$ where $\mathcal{Z}_{\mathbf{y}_i, \mathbf{x}_i}$ is the Koszul matrix factorization over $\text{Sym}(\mathbb{X}_i | \mathbb{X}'_i)$ with respect to the subsequences of \mathbf{x} and \mathbf{y} , which only consist of polynomials involving \mathbb{X}_i and \mathbb{X}'_i . $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ are the complementary subsequences, forming the Koszul complex $\mathcal{Z}_{\bar{\mathbf{y}}, \bar{\mathbf{x}}}$ over the subring of $\mathcal{R}_{\mathcal{B}} \otimes \mathcal{R}_{\mathcal{B}}$ generated by all alphabets except \mathbb{X}_i and \mathbb{X}'_i .

Proof. From the proof of Lemma 3.28 we see that $\bar{\mathbf{x}}$ is a regular sequence in $\mathcal{T}_{\mathcal{B}}$ and it clearly generates the ideal $\langle \mathbb{X}_j = \mathbb{X}'_j | j \neq i \rangle$. Thus we get the first isomorphism. For the second isomorphism note that after identifying $\mathbb{X}_j = \mathbb{X}'_j$ we can rewrite the defining relations in $\mathcal{T}_{\mathcal{B}}$ as

which impose $e_j(\mathbb{X}'_i) = e_j(\mathbb{X}_i)$ for $1 \leq j \leq k$, or in shorthand $\mathbb{X}'_i = \mathbb{X}_i$.

Lemma 3.30. $\text{MF}(\mathcal{B})$ has a simplification $\text{UMF}(\mathcal{B})$ over $\mathcal{V}_{\mathcal{B}}$.

$$(3.7) \quad \begin{array}{ccc} & \mathbb{Y}_1 & \mathbb{X}_1 \\ & \leftarrow & \rightarrow \\ & \mathbb{A} & \\ & \leftarrow & \rightarrow \\ & \mathbb{Y}_2 & \mathbb{X}_2 \end{array}$$

The Koszul matrix factorization associated to this web has positive entries $e_a(\mathbb{Y}_1 \sqcup \mathbb{A}) - e_a(\mathbb{X}_1)$ for $1 \leq a \leq |\mathbb{X}_1|$ and $e_b(\mathbb{Y}_2) - e_b(\mathbb{X}_2 \sqcup \mathbb{A})$ for $1 \leq b \leq |\mathbb{Y}_2| = |\mathbb{X}_2| + |\mathbb{A}|$, which includes the case $\mathbb{A} = \emptyset$. Now it is easy to see that a sequence of row operations as in Lemma 3.12 can be used to iteratively replace the first k positive entries $e_a(\mathbb{Y}_1 \sqcup \mathbb{A}) - e_a(\mathbb{X}_1)$ by $e_a(\mathbb{X}_1 \sqcup \mathbb{X}_2) - e_a(\mathbb{Y}_1 \sqcup \mathbb{Y}_2)$ for $1 \leq a \leq k$. Here we again use that all labels are $\geq k$. The new rows contain precisely the generators of the defining ideal of $\mathcal{V}_{\mathcal{B}}$ as positive entries. Since they form a regular sequence, we apply Proposition 3.13 and denote the result by $\text{UMF}(\mathcal{B})$. \square

As a Koszul matrix factorization, $\mathrm{UMF}(\mathcal{B})$ is free over $\mathcal{V}_{\mathcal{B}}$, so the proof of Lemma 3.28 together with Proposition 3.13 imply that $\mathrm{UMF}(\mathcal{B}) \otimes_{\mathcal{Z}_{\bar{\mathbf{y}}, \bar{\mathbf{x}}}}$ simplifies to $\mathrm{UMF}(\mathcal{B}) / \langle \mathbb{X}_j = \mathbb{X}'_j | j \neq i \rangle$, which is a $\mathrm{Sym}(\mathbb{X}_i | \mathbb{X}'_i) / \langle \mathbb{X}_i = \mathbb{X}'_i \rangle$ -module by Lemma 3.29. So we write it more compactly as $\mathrm{UMF}(\mathcal{B}) / \langle \mathbb{X}_j = \mathbb{X}'_j \rangle$. Further tensoring this with $\mathcal{Z}_{\mathbf{y}_i, \mathbf{x}_i}$ is equivalent to tensoring with $\mathcal{Z}_{\pi(\mathbf{y}_i), 0}$, where $\pi: \mathrm{Sym}(\mathbb{X}_i | \mathbb{X}'_i) \rightarrow \mathrm{Sym}(\mathbb{X}_i | \mathbb{X}'_i) / \langle \mathbb{X}_i = \mathbb{X}'_i \rangle$ is the obvious quotient map. We summarize:

Corollary 3.31. $\mathrm{MF}(\mathcal{L})$ has a simplification of the form:

$$(3.8) \quad \mathrm{MF}'(\mathcal{L}) := \mathrm{UMF}(\mathcal{B}) / \langle \mathbb{X}'_j = \mathbb{X}_j \rangle \otimes \mathcal{Z}_{\pi(\mathbf{y}_i), 0}$$

Further, note that we have an isomorphism of $\text{Sym}(\mathbb{X}_i)$ -modules:

$$H_{\text{tot}}(\mathcal{Z}_{\pi(\mathbf{y}_i), 0}) \cong H_{\text{tot}}\left(\frac{\text{Sym}(\mathbb{X}_i|\mathbb{X}'_i)}{\langle \mathbb{X}_i = \mathbb{X}'_i \rangle} \otimes_{\text{Sym}(\mathbb{X}_i|\mathbb{X}'_i)} \mathcal{Z}_{\mathbf{y}_i, \mathbf{x}_i}\right) = \text{KhR}^{\text{sl}_N}(\bigcirc^k) \cong H_k^N$$

So, the entries of $\pi(\mathbf{y}_i) \in \text{Sym}(\mathbb{X}_i)$ are generators for the defining ideal of H_k^N in $\text{Sym}(\mathbb{X}_i)$.

Definition 3.32. For a marked, labelled link diagram \mathcal{L} given as the closure of a braid diagram \mathcal{B} as specified in the introduction to this section, we define the *reduced complex* associated to \mathcal{L} as:

$$\overline{\text{MF}(\mathcal{L}(i))} := \text{UMF}(\mathcal{B}) / \langle \mathbb{X}_i = 0, \mathbb{X}_j = \mathbb{X}'_j \rangle$$

Remark 3.33. In order to prove the following two results we need to take a short technical detour. In their original work, Khovanov-Rozansky [29] and Wu [59] work with categories of \mathbb{Z}_2 -graded matrix factorizations. The matrix factorizations they associate to webs agree with the ones in this paper after reducing the \mathbb{Z} -grading modulo 2 and, in particular, their total homologies agree. However, the advantage of \mathbb{Z}_2 -graded matrix factorizations is that they admit a good notion of homotopy equivalence. Working in the corresponding homotopy category, Khovanov-Rozansky and Wu have proved matrix factorization analogues of many of the web isomorphisms that exist in the foam 2-category NgrFoam^\bullet . More generally, thanks to the 2-representation studied in [49, Section 4.4], we now know that all web isomorphisms in NgrFoam^\bullet imply the analogous relations in the corresponding homotopy 2-category of matrix factorizations. In the following lemma, we will use matrix factorization versions of the web relations employed in web evaluation algorithm from Proposition 4.5, and we will use that these relations preserve the module structure with respect to alphabet assigned to the marked edge.

Lemma 3.34. *There is a homotopy equivalence of complexes over the homotopy category of \mathbb{Z}_2 -graded matrix factorizations over $\text{Sym}(\mathbb{X}_i|\mathbb{X}'_i)$*

$$(\text{MF}(\mathcal{L}), d_v) \rightarrow (C(\mathcal{L}), d_v),$$

where the target is a chain complex of whose chain groups are direct sums of terms $\text{MF}(\bigcirc^k)$. Upon taking total homology, this induces a homotopy equivalence of complexes of $\text{Sym}(\mathbb{X}_i)$ -modules:

$$(H_{\text{tot}}(\text{MF}(\mathcal{L})), d_v^*) \rightarrow (H_{\text{tot}}(C(\mathcal{L})), d_v^*).$$

Proof. Consider the $(1, 1)$ -tangle diagram \mathcal{T} obtained by cutting open the marked edge in \mathcal{L} , and its associated complex of matrix factorizations

$$\text{MF}(\mathcal{T}) := \text{MF}(\mathcal{B}) \otimes \mathcal{Z}_{\bar{\mathbf{y}}, \bar{\mathbf{x}}}.$$

The web evaluation algorithm from Proposition 4.5 together with Remark 3.33 imply that the complex $\text{MF}(\mathcal{T})$ of matrix factorization is homotopy equivalent (in the category of chain complexes over the homotopy category of \mathbb{Z}_2 -graded matrix factorizations over $\text{Sym}(\mathbb{X}_i|\mathbb{X}'_i)$) to a complex $O(\mathcal{L})$ whose chain groups are direct sums of the matrix factorizations $\mathcal{Z}_{\mathbf{y}_i, -\mathbf{x}_i}$ assigned to the trivial k -labelled 1-strand tangle. After closing off the marked edge by tensoring with $\mathcal{Z}_{\mathbf{y}_i, \mathbf{x}_i}$, we obtain the desired homotopy equivalence:

$$\text{MF}(\mathcal{L}) = \text{MF}(\mathcal{T}) \otimes \mathcal{Z}_{\mathbf{y}_i, \mathbf{x}_i} \rightarrow O(\mathcal{L}) \otimes \mathcal{Z}_{\mathbf{y}_i, \mathbf{x}_i} = C(\mathcal{L}).$$

This preserves the $\text{Sym}(\mathbb{X}_i|\mathbb{X}'_i)$ -module structure on the matrix factorizations and induces a homotopy equivalence of $\text{Sym}(\mathbb{X}_i|\mathbb{X}'_i)/\langle \mathbb{X}_i = \mathbb{X}'_i \rangle \cong \text{Sym}(\mathbb{X}_i)$ -modules after taking total homology. \square

In the following, let $V := \text{Sym}(\mathbb{X}_i)/\langle \mathbb{X}_i = 0 \rangle$ and $V' := \text{Sym}(\mathbb{X}'_i)/\langle \mathbb{X}'_i = 0 \rangle$.

Proposition 3.35. *There is a homotopy equivalence of chain complexes with respect to d_v^* :*

$$V \otimes H_{\text{tot}}(\text{MF}(\mathcal{L})) \sim H_{\text{tot}}(\overline{\text{MF}(\mathcal{L}(i))})$$

Proof. For the proof we express $H_{tot}(V \otimes \text{MF}(\mathcal{L}))$ in two different ways. In the case of $\mathcal{L} = \bigcirc^k$ we get:

$$\begin{aligned} H_{tot}(V \otimes \text{MF}(\bigcirc^k)) &= H_{tot}(V \otimes \mathcal{Z}_{\mathbf{y}_i, -\mathbf{x}_i} \otimes \mathcal{Z}_{\mathbf{y}_i, \mathbf{x}_i}) \cong H_{tot}((V \otimes V') \otimes \mathcal{Z}_{\mathbf{y}_i, \mathbf{x}_i}) \\ &= H_{tot}((V \otimes V') \otimes \mathcal{Z}_{0,0}) \cong V \otimes H^*(S^1)^{\otimes k} \\ &\cong V \otimes H_{tot}(\text{MF}(\bigcirc^k)) \otimes H^*(S^1)^{\otimes k} \end{aligned}$$

Here the first isomorphism is provided by Proposition 3.13. More generally, we consider a complex $C(\mathcal{L}) = O(\mathcal{L}) \otimes \mathcal{Z}_{\mathbf{y}_i, \mathbf{x}_i}$ – as in the proof of Lemma 3.34 – with vertical differential supported on $O(\mathcal{L})$, whose chain groups are direct sums of $\mathcal{Z}_{\mathbf{y}_i, -\mathbf{x}_i}$. Analogously as in the unknot case, we can simplify the direct summands $V \otimes \mathcal{Z}_{\mathbf{y}_i, -\mathbf{x}_i}$ of the chain groups of $V \otimes O(\mathcal{L})$ to terms of the form $V \otimes V'$. After tensoring with the single-strand closure $\mathcal{Z}_{\mathbf{y}_i, \mathbf{x}_i}$ and taking total homology, we get an isomorphism of chain complexes

$$H_{tot}(V \otimes C(\mathcal{L})) \cong V \otimes H_{tot}(C(\mathcal{L})) \otimes H^*(S^1)^{\otimes k}$$

in which the vertical differential acts trivially on the tensor factor $H^*(S^1)^{\otimes k}$ contributed by the single-strand closure $\mathcal{Z}_{\mathbf{y}_i, \mathbf{x}_i}$. Using the homotopy equivalences from Lemma 3.34, which preserve the $\text{Sym}(\mathbb{X}_i)$ -module structures before and after taking total homology, we now get:

$$\begin{aligned} (3.9) \quad H_{tot}(V \otimes \text{MF}(\mathcal{L})) &\sim H_{tot}(V \otimes C(\mathcal{L})) \cong V \otimes H_{tot}(C(\mathcal{L})) \otimes H^*(S^1)^{\otimes k} \\ &\sim V \otimes H_{tot}(\text{MF}(\mathcal{L})) \otimes H^*(S^1)^{\otimes k} \end{aligned}$$

On the other hand, we can tensor (3.8) with V and take H_{tot} homology to get:

$$\begin{aligned} (3.10) \quad H_{tot}(V \otimes \text{MF}(\mathcal{L})) &\cong H_{tot}(V \otimes \text{MF}'(\mathcal{L})) \\ &= H_{tot}\left(V \otimes \frac{\text{UMF}(\mathcal{B})}{\langle \mathbb{X}_j = \mathbb{X}'_j \rangle} \otimes \mathcal{Z}_{\pi(\mathbf{y}_i), 0}\right) \cong H_{tot}(\overline{\text{MF}(\mathcal{L}(i))}) \otimes H^*(S^1)^{\otimes k}. \end{aligned}$$

On the latter complex, the vertical differential acts trivially on the tensor factor $H^*(S^1)^{\otimes k}$, which is again the contribution of the closure of the marked edge. Since the homotopy category of graded chain complexes of \mathbb{C} -vector spaces is Krull-Schmidt, we can cancel the factors of $H^*(S^1)^{\otimes k}$ in (3.9) and (3.10), which produces the desired homotopy equivalence. \square

Corollary 3.36. *The reduced Khovanov-Rozansky homology is computed by the reduced complex:*

$$\overline{\text{KhR}}^{\text{sl}_N}(\mathcal{L}(i)) \cong H_*(H_{tot}(\overline{\text{MF}(\mathcal{L}(i))}), d_v^*)$$

Proof. Analogously as in [42, Proof of Theorem 4.12], it follows from the 2-representation of foams on matrix factorizations that $(H_{tot}(\text{MF}(\mathcal{L})), d_v^*)$ and $\text{graut}(\llbracket \mathcal{L} \rrbracket^{\text{sl}_N})$ are isomorphic as chain complexes of H_k^N -modules. The reduced homology is then obtained from tensoring the chain groups of these complexes with the H_k^N -module $\langle \pi_k^N \rangle$ before taking vertical homology. Clearly, this is equivalent to tensoring with the isomorphic H_k^N -module $V \otimes H_k^N$. Now the claim follows from Proposition 3.35. \square

Definition 3.37. Let $\mathcal{L}(i)$ be the closure of a balanced labelled braid diagram \mathcal{B} with a marked lower boundary point o_i of minimal label. Then we define the reduced colored HOMFLY-PT homology of \mathcal{L} reduced at i as

$$\overline{\text{KhR}}^\infty(\mathcal{L}(i)) := H_*(H^+(\overline{\text{MF}(\mathcal{L}(i))}), d_v^*).$$

Let $\mathcal{W}_{\mathcal{B}} := \mathcal{V}_{\mathcal{B}} / \langle \mathbb{X}_j = \mathbb{X}'_j | j \neq i \rangle = \mathcal{V}_{\mathcal{B}} / \langle \mathbb{X}_j = \mathbb{X}'_j \rangle$, which is still isomorphic to a polynomial ring containing a tensor factor $\text{Sym}(\mathbb{X}_i)$. Thus we can write $\mathcal{W}_{\mathcal{B}} \cong \mathcal{W}_{\mathcal{B}} / \langle \mathbb{X}_i = 0 \rangle \otimes \text{Sym}(\mathbb{X}_i)$. In the following, we consider $\text{UMF}(\mathcal{B}) / \langle \mathbb{X}_j = \mathbb{X}'_j \rangle$ as a (vertical) complex of matrix factorizations over $\mathcal{W}_{\mathcal{B}} / \langle \mathbb{X}_i = 0 \rangle$.

Lemma 3.38. *As a complex of matrix factorizations over $\mathcal{W}_{\mathcal{B}}/\langle \mathbb{X}_i = 0 \rangle$, the reduced complex $\overline{\text{MF}}(\mathcal{L}(i))$ is homotopy equivalent to the iterated mapping cone $\text{UMF}(\mathcal{B})/\langle \mathbb{X}_j = \mathbb{X}'_j \rangle \otimes \mathcal{Z}_{\mathbb{X}_i}$. Here $\mathcal{Z}_{\mathbb{X}_i}$ is the (vertical) Koszul complex over $\text{Sym}(\mathbb{X}_i)$ with respect to the sequence consisting of $e_l(\mathbb{X}_i)$ for $1 \leq l \leq k$.*

Proof. For $0 \leq l \leq k$ define $\mathcal{W}_{\mathcal{B}}^l := \mathcal{W}_{\mathcal{B}}/\langle e_x(\mathbb{X}_i) = 0 | x > l \rangle$ and $\overline{\text{MF}}(\mathcal{L}(i))_l := \mathcal{W}_{\mathcal{B}}^l \otimes \text{UMF}(\mathcal{B})$, which is a free matrix factorizations over $\mathcal{W}_{\mathcal{B}}^l$. As special cases we have $\overline{\text{MF}}(\mathcal{L}(i))_k = \text{UMF}(\mathcal{B})/\langle \mathbb{X}_j = \mathbb{X}'_j \rangle$ and $\overline{\text{MF}}(\mathcal{L}(i))_0 = \overline{\text{MF}}(\mathcal{L}(i))$. Now Rasmussen's one-variable argument from [47, Lemma 5.15] shows that $\overline{\text{MF}}(\mathcal{L}(i))_l$ is homotopy equivalent, as a complex of matrix factorizations over $\mathcal{W}_{\mathcal{B}}^l$, to $\overline{\text{MF}}(\mathcal{L}(i))_{l+1} \otimes \mathcal{Z}_{e_l(\mathbb{X}_i)}$. The same holds over sub-rings $\mathcal{W}_{\mathcal{B}}^{l'}$ with $l' < l$, and so we can simultaneously trade quotienting out all generators $e_l(\mathbb{X}_i)$ against tensoring with the Koszul complex $\mathcal{Z}_{\mathbb{X}_i}$. \square

After taking positive homology, we get a homotopy equivalence of vertical complexes

$$(3.11) \quad H^+(\overline{\text{MF}}(\mathcal{L}(i))) \sim H^+(\text{UMF}(\mathcal{B})/\langle \mathbb{X}_j = \mathbb{X}'_j \rangle) \otimes \mathcal{Z}_{\mathbb{X}_i}$$

Lemma 3.39. *$\overline{\text{KhR}}^\infty(\mathcal{L}(i))$ is isomorphic to $\overline{\text{KhR}}^\infty(\mathcal{L}(i'))$ if the marked points o_i and $o_{i'}$ lie on the same component of \mathcal{L} .*

Proof. Equation (3.11) shows that $\overline{\text{KhR}}^\infty(\mathcal{L}(i))$ is the homology of an iterated mapping cone on endomorphisms of the complex $H^+(\text{UMF}(\mathcal{B})/\langle \mathbb{X}_j = \mathbb{X}'_j \rangle)$, which does not depend on i . Since cones on homotopic chain maps are isomorphic, it suffices to show that the endomorphism of $H^+(\text{UMF}(\mathcal{B})/\langle \mathbb{X}_j = \mathbb{X}'_j \rangle)$ given by multiplying by $e_l(\mathbb{X}_i)$ is homotopic to the endomorphism given by multiplying by $e_l(\mathbb{X}_{i'})$. In fact, it is enough to check this homotopy between endomorphisms locally, i.e. on the complex associated to a single crossing, where the alphabets \mathbb{X}_i and $\mathbb{X}_{i'}$ are assigned to opposite sides of a single strand. This has been done by Rose and the author in [49, Proposition 60] in the framework of webs and foams, and an analogous proof works in the present setting. \square

As a by-product we get that the $\mathcal{S}_{\mathcal{B}}$ action on $\overline{\text{KhR}}^\infty(\mathcal{L}(i))$ factors through the quotient $\mathcal{S}_{\mathcal{B}}/\langle \mathbb{Y}_j = \mathbb{Y}_{j'}, \mathbb{X}_i = 0 \rangle$ where we identify alphabets $\mathbb{Y}_j, \mathbb{Y}_{j'}$ which belong to the same link component. In particular, all alphabets on the component containing the marked point o_i act by zero.

Corollary 3.40. *The triply-graded vector space $\overline{\text{KhR}}^\infty(\mathcal{L}(i))$ is independent of the diagram of the labelled link with marked component specified by i .*

Proof. From Corollary 3.31 we get an isomorphism of chain complexes with respect of d_v^* :

$$(3.12) \quad (H^+(\text{MF}(\mathcal{L})), d_v^*) \cong (H^+(\text{UMF}(\mathcal{B})/\langle \mathbb{X}_j = \mathbb{X}'_j \rangle), d_v^*) \otimes H^*(S^1)^{\otimes k}$$

and this decomposition does not depend on i , only on its label. Since $H^+(\text{MF}(\mathcal{L}))$ is invariant under Reidemeister 2 and 3 and Markov moves up to homotopy, so is $H^+(\text{UMF}(\mathcal{B})/\langle \mathbb{X}_j = \mathbb{X}'_j \rangle)$. The same is true for this complex tensored with $\mathcal{Z}_{\mathbb{X}_i}$, as we may assume that the alphabet \mathbb{X}_i is associated to an edge not participating in the move (see proof of Lemma 3.39). \square

Remark 3.41. $H_*(H^+(\text{UMF}(\mathcal{B})/\langle \mathbb{X}_j = \mathbb{X}'_j \rangle), d_v^*)$ is a triply-graded link invariant which is the colored generalization of what Rasmussen calls the *middle HOMFLY homology*, [47, Definition 2.9]. It carries exactly the same amount of information as $\text{KhR}^\infty(\mathcal{L})$.

Remark 3.42. In the uncolored case, Rasmussen shows that $\text{KhR}^\infty(\mathcal{L}) \cong \overline{\text{KhR}}^\infty(\mathcal{L}(i)) \otimes \text{KhR}^\infty(\bigcirc)$, see [47, Section 2.8]. The key ingredient for this is an isomorphism $\text{UMF}(\mathcal{B})/\langle \mathbb{X}_j = \mathbb{X}'_j \rangle \cong \overline{\text{MF}}(\mathcal{L}(i)) \otimes \text{Sym}(\mathbb{X}_i)$ of double complexes with respect to d and d_+ . We suspect that this also holds in the colored case, but do not know a proof of this fact.

Theorem 3.43. *There is a spectral sequence*

$$\overline{\text{KhR}}^\infty(\mathcal{L}(i)) \rightsquigarrow \overline{\text{KhR}}^{\mathfrak{sl}_N}(\mathcal{L}(i))$$

whose k^{th} differential lowers horizontal grading by k , homological grading by k and it increases the internal grading by $2kN$.

Proof. Analogously as in the unreduced situation of Theorem 3.22, we have a vertical complex of matrix factorizations $\overline{\text{MF}}(\mathcal{L}(i))$ with potential zero, from which HOMFLY-PT as well as \mathfrak{sl}_N homologies can be recovered:

$$\begin{aligned} \overline{\text{KhR}}^\infty(\mathcal{L}(i)) &= H_*(H^+(\overline{\text{MF}}(\mathcal{L}(i))), d_v^*) \\ \overline{\text{KhR}}^{\mathfrak{sl}_N}(\mathcal{L}(i)) &\cong H_*(H_{\text{tot}}(\overline{\text{MF}}(\mathcal{L}(i))), d_v^*) \end{aligned}$$

The proof now proceeds as for Theorem 3.22, or its uncolored predecessor in [47]. The necessary reduced analogue of Lemma 3.23 can be proved by using the web evaluation algorithm introduced in Proposition 4.5, which allows to keep the marked edge undisturbed in the simplification process. Further, it is easy to check by hand that the additional bigon relation (4.1) holds in the categorified framework and that it causes a homogeneous shift in horizontal grading in all webs appearing in the algorithm (just like removing circles of labels that sum up $\sum o_i$). \square

Lemma 3.44. *For any web W appearing in $\llbracket \mathcal{B} \rrbracket$, the singular Soergel bimodule $\text{Dec}(W)$ is finitely generated over $\mathcal{S}_{\mathcal{B}}$.*

Proof. This would be obvious from the definition of $\text{Dec}(W)$ if $\mathcal{S}_{\mathcal{B}}$ would contain an alphabet for every web edge of W . However, W contains additional web edges on top of the ones already present as arcs of the diagram \mathcal{B} – namely the ones introduced in crossing resolutions. It remains to show that all decorations on the corresponding bad facets can be expressed as $\mathcal{S}_{\mathcal{B}}$ -linear combination of only a finite number of basis decorations. For this, we consider a single crossing site:

$$(3.13) \quad \begin{array}{ccccc} \mathbb{Y}'_1 & & \mathbb{W}_3 & & \mathbb{Y}_1 \\ \downarrow & \swarrow & & \searrow & \downarrow \\ \mathbb{W}_2 & & & & \mathbb{W}_1 \\ \downarrow & \swarrow & & \searrow & \downarrow \\ \mathbb{Y}'_2 & & \mathbb{W}_4 & & \mathbb{Y}_2 \end{array}$$

Here, the edges labelled with the alphabets \mathbb{Y}_i come from arcs in the diagram \mathcal{B} and the edges labelled by \mathbb{W}_i alphabets are new. After eliminating \mathbb{W}_4 , we see that the singular Soergel bimodule associated to this web is isomorphic to:

$$(3.14) \quad \frac{\text{Sym}(\mathbb{Y}'_1|\mathbb{W}_2|\mathbb{W}_3)}{(\mathbb{Y}'_1 = \mathbb{W}_2 \sqcup \mathbb{W}_3)} \otimes_{\text{Sym}(\mathbb{W}_2|\mathbb{W}_3)} \frac{\text{Sym}(\mathbb{Y}_2|\mathbb{Y}'_2|\mathbb{W}_1|\mathbb{W}_2|\mathbb{W}_3)}{(\mathbb{Y}'_2 \sqcup \mathbb{W}_2 = \mathbb{Y}_2 \sqcup \mathbb{W}_1)} \otimes_{\text{Sym}(\mathbb{W}_1|\mathbb{W}_3)} \frac{\text{Sym}(\mathbb{W}_1|\mathbb{W}_3|\mathbb{Y}_1)}{(\mathbb{W}_1 \sqcup \mathbb{W}_3 = \mathbb{Y}_1)}$$

It is well known that $\text{Sym}(\mathbb{W}_2|\mathbb{W}_3)$ is a free $\text{Sym}(\mathbb{W}_2 \sqcup \mathbb{W}_3)$ -module of rank $\binom{|\mathbb{W}_2 \sqcup \mathbb{W}_3|}{|\mathbb{W}_2|}$ with a basis given by Schur polynomials $\pi_\lambda^{\mathbb{W}_2}$ with $\lambda \in P(|\mathbb{W}_2|, |\mathbb{W}_3|)$. Thus, the left tensor factor is of finite rank over $\text{Sym}(\mathbb{Y}'_1)$. The case of the right tensor factor is completely analogous and, consequently, the total bimodule is finitely generated over $\text{Sym}(\mathbb{Y}_1|\mathbb{Y}_2|\mathbb{Y}'_1|\mathbb{Y}'_2)$. \square

Proposition 3.45. *For a labelled knot \mathcal{K} , the reduced colored HOMFLY-PT homology $\overline{\text{KhR}}^\infty(\mathcal{K})$ is finite dimensional.*

Proof. From Lemma 3.44 we see that $H^+(\text{MF}(\mathcal{L})) \cong H^+(\text{Dec}(\mathcal{B}) \otimes \mathcal{Z}_{\mathbf{y}, \mathbf{x}})$ is finitely generated over $\mathcal{S}_{\mathcal{B}}$ and so is the iterated mapping cone $H^+(\text{MF}(\mathcal{L})) \otimes \mathcal{Z}_{\mathbb{X}_i}$. From (3.12) and Lemma 3.38 we see:

$$H^+(\text{MF}(\mathcal{L})) \otimes \mathcal{Z}_{\mathbb{X}_i} \cong H^+(\text{UMF}(\mathcal{B})/\langle \mathbb{X}_j = \mathbb{X}'_j \rangle) \otimes H^*(S^1)^{\otimes k} \otimes \mathcal{Z}_{\mathbb{X}_i} \sim H^+(\overline{\text{MF}}(\mathcal{L}(i))) \otimes H^*(S^1)^{\otimes k}$$

The vertical homology of this iterated cone is isomorphic as a triply-graded vector space to a direct sum of 2^k copies of $\overline{\text{KhR}}^\infty(\mathcal{K})$ and we shall argue that it is finite-dimensional. By basic homological algebra, the generators $\text{Sym}(\mathbb{X}_i)$ act null-homotopically on the iterated cone and by zero on its vertical homology. By the proof of Lemma 3.39, the same holds for all alphabets in $\mathcal{S}_{\mathcal{B}}$ in the case of a knot $\mathcal{L} = \mathcal{K}$. The vertical homology of the iterated cone is a finitely generated $\mathcal{S}_{\mathcal{B}}$ -module, but all variables act by zero, so it is finitely generated over \mathbb{C} . Consequently, its tensor factor $\overline{\text{KhR}}^\infty(\mathcal{K})$ is finite-dimensional. \square

Theorem 3.46. *For a labelled knot \mathcal{K} , the reduced colored HOMFLY-PT homology $\overline{\text{KhR}}^\infty(\mathcal{K})$ is a stabilization of reduced colored \mathfrak{sl}_N homologies $\overline{\text{KhR}}^{\mathfrak{sl}_N}(\mathcal{K})$ for large N . More precisely:*

$$(3.15) \quad \bigoplus_{\substack{i+2Nj=I \\ h-j=J}} \overline{\text{KhR}}_{i,j,h}^\infty(\mathcal{K}) \cong \overline{\text{KhR}}_{I,J}^{\mathfrak{sl}_N}(\mathcal{K})$$

Proof. We know that $\overline{\text{KhR}}^\infty(\mathcal{K})$ is finite-dimensional and, thus, supported in finitely many internal gradings. Since the k^{th} differential in the spectral sequence from Theorem 3.43 increases the internal grading by $2Nk$, all higher differentials must be trivial for large N . So the spectral sequence, which is known to compute \mathfrak{sl}_N homology, must have already converged on the first page, which is precisely the grading-collapsed version of HOMFLY-PT homology displayed in (3.15). \square

Corollary 3.47. (Refined exponential growth for HOMFLY-PT homology) *There exist spectral sequences*

$$(3.16) \quad \overline{\text{KhR}}^\infty(\mathcal{K}^k) \rightsquigarrow \left(\overline{\text{KhR}}^\infty(\mathcal{K}^1) \right)^{\otimes k}$$

$$(3.17) \quad \overline{\text{KhR}}^\infty(\mathcal{K}^k) \rightsquigarrow \overline{\text{KhR}}^\infty(\mathcal{K}^{k-1}) \otimes \overline{\text{KhR}}^\infty(\mathcal{K}^1)$$

which preserve the homological grading. In particular, the colored HOMFLY-PT homologies of a knot grow at least exponentially in color.

Proof. Immediate from Theorem 3.46 and the spectral sequences from Corollary 2.31 for $N \gg 0$. \square

Corollary 3.48. (Color-reducing spectral sequences) *For $k \geq h$ there is a spectral sequence*

$$(3.18) \quad \overline{\text{KhR}}^\infty(\mathcal{K}^k) \rightsquigarrow \overline{\text{KhR}}^\infty(\mathcal{K}^h)$$

which preserves the homological grading (up to a shift).

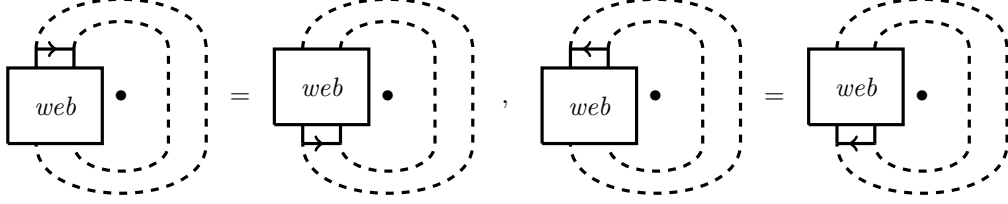
Proof. Immediate from Theorem 3.46 and the spectral sequences from Corollary 2.33 for $N \gg 0$. \square

4. APPENDIX ON WEB EVALUATION

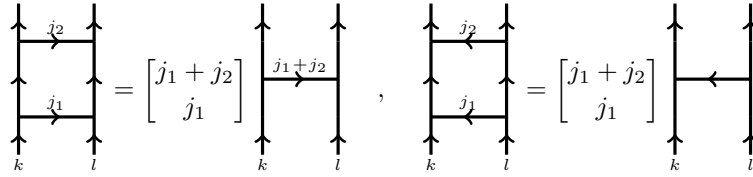
This appendix contains an algorithm for the evaluation of \mathfrak{sl}_N webs with two boundary points. For this, we work in the decategorified framework of \mathfrak{sl}_N webs, see e.g. Cautis-Kamnitzer-Morrison's paper [8] or Murakami-Ohtsuki-Yamada's earlier work [40]. First we recall the annular web evaluation algorithm of Queffelec-Rose.

Lemma 4.1. [43, Lemma 5.2] *The annular closure of an \mathfrak{sl}_N (ladder) web can be rewritten as a $\mathbb{Z}[q^{\pm 1}]$ -linear combination of nested, labelled essential circles using only the following four relations:*

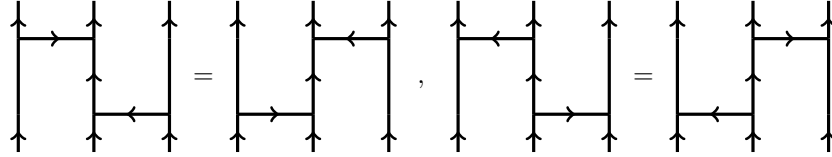
(1) *Sliding ladder rungs around the annulus*



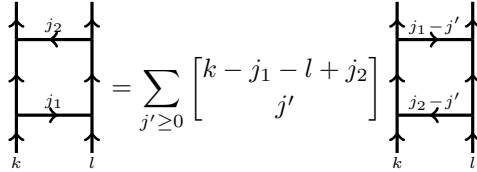
(2) *Ladder rung combination*



(3) *Frobenius relations*



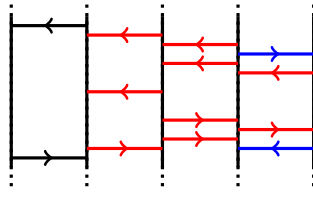
(4) *Square switch relations*



Here we write $\begin{bmatrix} a \\ b \end{bmatrix}$ for the q -binomial coefficients and the webs in these relations have been rotated to make edges upward (or not-downward) oriented and they have been isotoped into the *ladder form* consisting of a fixed number of *uprights* and *ladder rungs* between them. We recall the explicit algorithm, which is used to prove this result, as we will need a modification of it later.

Proof. [43] At the start, and then throughout, the algorithm combines all possible adjacent ladder rungs pointing in the same direction using relations (2) and (3). As a first goal, the algorithm aims to free the leftmost upright from the rest of the diagram by repeatedly applying relation (4) to pairs of rungs adjacent to it. If possible, this has the effect of increasing a label on the leftmost upright or it allows to reduce the number of rungs touching it, which shows that this algorithm eventually terminates and succeeds in splitting of the leftmost upright. It suffices to argue that relation (4) can always be applied to a pair of rungs connected to the leftmost upright (except in the trivial case when the leftmost upright is already disconnected). Clearly, one can always find a leftwards rung above a rightward rung by relation (1), however, they might be separated by *trapped rungs* further to the right in the web. We

reproduce an example given in [43]:



Here the black rungs are the chosen ones, which are separated by trapped rungs colored red. The blue rungs are not trapped and can be moved outside the local picture by applying relation (3). Queffelec and Rose show that all trapped rungs can be untrapped by applying relations (3) and (4), without moving the chosen pair of rungs on the left. We shall call this:

The untrapping subroutine: Its strategy is to move the lowest leftward pointing trapped rung downwards until it becomes untrapped. This is possible because below this leftward rung, there can only be rightward pointing trapped rungs, which do not obstruct the downward passage of the leftward rung. To see this, we distinguish three cases:

- The rightward rung appears below the leftward rung, but they are adjacent to four different uprights. Then the leftward rung slides down past the other rung by a planar isotopy.
- The rightward rung appears below the leftward rung and they share exactly one upright. Then the leftward rung can be moved down past the rightward rung by relation (3).
- The rightward rung appears immediately below the leftward rung. Then we apply relation (4) to switch them, which creates summands in which the leftwards rung is below the corresponding rightward rung (this includes the possible case of label 0 on a rung, in which case we think of it as erased).

After all leftward trapped rungs have been moved downwards and outside the local picture, all rightwards (formerly) trapped rungs can be moved upwards by relation (3). Then, the chosen rungs adjacent to the leftmost upright are no longer separated and relation (4) can be applied, as desired.

Once the leftmost upright is disconnected, the algorithm is recursively applied to the rest of the diagram. \square

Definition 4.2. We call an \mathfrak{sl}_N ladder web of *type EF* if each leftward-oriented ladder rung appears under all rightward-oriented ladder rungs adjacent to the same two uprights. Analogously, we call a web an \mathfrak{sl}_N ladder web of *type FE* if rightward rungs appear below leftward rungs in each *column*.

We need a generalization of the untrapping subroutine

Lemma 4.3. The resorting subroutine. Any \mathfrak{sl}_N ladder web can be rewritten as a $\mathbb{Z}[q^{\pm 1}]$ -linear combination of ladder webs of type EF (FE), using only the relations (2), (3) and (4).

Proof. We only describe an algorithm for resorting to EF webs. The other case is completely analogous; instead of the relations (4), it uses the relations (4'), which can be deduced from (4):

(4') Other square switch relations

$$\begin{array}{c} \uparrow \\ \text{---} j_2 \text{---} \\ \uparrow \\ \text{---} j_1 \text{---} \\ \uparrow \\ \text{---} k \end{array} \quad \begin{array}{c} \uparrow \\ \text{---} l \end{array} = \sum_{j' \geq 0} \begin{bmatrix} l - j_1 - k + j_2 \\ j' \end{bmatrix} \begin{array}{c} \uparrow \\ \text{---} j_1 \text{---} j' \text{---} \\ \uparrow \\ \text{---} j_2 \text{---} j' \text{---} \\ \uparrow \\ \text{---} k \end{array} \quad \begin{array}{c} \uparrow \\ \text{---} l \end{array}$$

Also, we have the choice of starting from the leftmost or from the rightmost upright in the ladder web diagram; we describe the former. Without further mention, we assume that the algorithm applies the relations (2) whenever possible.

Suppose that adjacent to the leftmost upright, there is a leftward rung immediately above a rightward rung. Then the untrapping subroutine shows that all rungs that are trapped between our chosen rungs can be moved away, modulo relations applied further to the right in the web. The result is a configuration, in which our chosen rungs form a square that can be switched via relation (4). In doing so, the leftward rung moves down past the rightward rung. As long as there are leftward rungs above rightward rungs in the first column, this procedure can be iterated. Finally, it terminates in a linear combination of webs in which the rungs in the first column satisfy the EF condition.

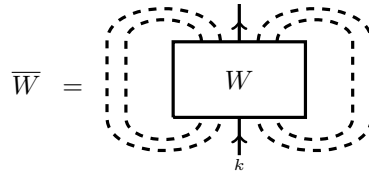
Now suppose that the rungs in the first $x - 1$ columns on the left are already sorted to satisfy the EF condition. Suppose there is a leftward rung immediately above a rightward rung in the x^{th} column. By the untrapping routine we may assume that there are no trapped further to the right in the web. Since the web is already of type EF in the $(x - 1)^{\text{th}}$ column, there are also no trapped rungs on the left. Thus, relation (4) can be applied to move the leftward rung down past the rightward rung. This process does not change anything to the left of the x^{th} column (modulo relation (3)) and it terminates after finitely many iterations in a linear combination of webs, in which the rungs in the first x columns satisfy the EF condition. \square

Remark 4.4. Via quantum skew Howe duality [8], the existence of a resorting subroutine follows from the triangular decomposition of quantum \mathfrak{gl}_m .

In the following, we need another bigon relation which is satisfied for \mathfrak{sl}_N webs in \mathbb{R}^2 :

$$(4.1) \quad \begin{array}{c} \text{leftward rung } j \text{ above rightward rung } k \end{array} = \begin{bmatrix} N - k \\ j \end{bmatrix} \begin{array}{c} \text{rightward rung } k \end{array} = \begin{array}{c} \text{rightward rung } k \text{ above leftward rung } j \end{array}$$

Proposition 4.5. *Let W be a balanced \mathfrak{sl}_N ladder web with a marked boundary point of label k and consider the partial closure \overline{W} of all unmarked boundary strands:*



Then \overline{W} can be rewritten as a $\mathbb{Z}[q^{\pm 1}]$ -linear combination of the k -labelled web edge using only the four relations from Lemma 4.1 and the additional relation (4.1). The latter is used in the following form (and its horizontally reflected version):

(5) *Partial closure relations*

$$\begin{array}{c} \text{partial closure of } W \end{array} = \begin{bmatrix} N - \sum j_i \\ l \end{bmatrix} \begin{array}{c} \text{web edge } k \end{array}$$

Proof. We describe an explicit algorithm for the case that W has closed strands on both sides of the marked upright; the other cases are easier. First we apply the resorting subroutine from Lemma 4.3 to replace W by a linear combination of webs of type FE. Then the relation (5) can be applied to the rightmost upright to remove its closure. We interpret the resulting webs as ladder webs whose rightmost upright has label 0 at the bottom and the top.

In the next step, we resort all webs to EF type. This has the effect of removing any interaction with the rightmost upright and reduces the problem to ladder webs with one fewer upright. Also, having resorted, we can now use the reflected version of relation (5) to remove the closure of the leftmost upright. After finitely many steps of resorting between types FE and EF, and removing closures via the relations (5), the algorithm terminates in a multiple of the marked edge. \square

REFERENCES

- [1] J. A. Baldwin, A. S. Levine, and S. Sarkar. Khovanov homology and knot Floer homology for pointed links. [ArXiv e-prints](#), December 2015.
- [2] Dror Bar-Natan. Khovanov’s homology for tangles and cobordisms. *Geom. Topol.*, 9:1443–1499, 2005.
- [3] Hanno Becker. Khovanov-Rozansky homology via Cohen-Macaulay approximations and Soergel bimodules. [ArXiv e-prints](#), May 2011.
- [4] Christian Blanchet. An oriented model for Khovanov homology. *J. Knot Theory Ramifications*, 19(2):291–312, 2010.
- [5] Nils Carqueville and Daniel Murfet. Adjunctions and defects in Landau-Ginzburg models. [ArXiv e-prints](#), August 2012.
- [6] Nils Carqueville and Daniel Murfet. Computing Khovanov-Rozansky homology and defect fusion. *Algebr. Geom. Topol.*, 14(1):489–537, 2014.
- [7] Sabin Cautis. Clasp technology to knot homology via the affine Grassmannian. *Math. Ann.*, 363(3-4):1053–1115, 2015.
- [8] Sabin Cautis, Joel Kamnitzer, and Scott Morrison. Webs and quantum skew Howe duality. *Math. Ann.*, 360(1-2):351–390, 2014.
- [9] Joseph Chuang and Raphaël Rouquier. Derived equivalences for symmetric groups and \mathfrak{sl}_2 -categorification. *Ann. of Math. (2)*, 167(1):245–298, 2008.
- [10] David Clark. Functoriality for the \mathfrak{su}_3 Khovanov homology. *Algebr. Geom. Topol.*, 9(2):625–690, 2009.
- [11] N. Dowlin. Knot Floer Homology and Khovanov-Rozansky Homology for Singular Links. [ArXiv e-prints](#), November 2015.
- [12] Nathan M. Dunfield, Sergei Gukov, and Jacob Rasmussen. The superpolynomial for knot homologies. *Experiment. Math.*, 15(2):129–159, 2006.
- [13] A. P. Ellis, I. Petkova, and V. Vértési. Quantum $\mathfrak{gl}(1|1)$ and tangle Floer homology. [ArXiv e-prints](#), October 2015.
- [14] Hiroyuki Fuji, Sergei Gukov, and Piotr Sułkowski. Super- A -polynomial for knots and BPS states. *Nuclear Phys. B*, 867(2):506–546, 2013.
- [15] Hiroyuki Fuji, Sergei Gukov, Piotr Sułkowski, and Hidetoshi Awata. Volume conjecture: refined and categorified. *Adv. Theor. Math. Phys.*, 16(6):1669–1777, 2012.
- [16] Stavros Garoufalidis. On the characteristic and deformation varieties of a knot. In *Proceedings of the Casson Fest*, volume 7 of *Geom. Topol. Monogr.*, pages 291–309 (electronic). Geom. Topol. Publ., Coventry, 2004.
- [17] Stavros Garoufalidis. The colored HOMFLY polynomial is q-holonomic. [ArXiv e-prints](#), November 2012.
- [18] Eugene Gorsky, Sergei Gukov, and Marko Stošić. Quadruply-graded colored homology of knots. [ArXiv e-prints](#), April 2013.
- [19] J. Grant. A categorification of the skew Howe action on a representation category of $U_q(\mathfrak{gl}(m|n))$. [ArXiv e-prints](#), April 2015.
- [20] Sergei Gukov, Satoshi Nawata, Ingmar Saberi, Marko Stošić, and Piotr Sułkowski. Sequencing BPS Spectra. [ArXiv e-prints](#), December 2015.
- [21] Sergei Gukov and Ingmar Saberi. Lectures on knot homology and quantum curves. In *Topology and field theories*, volume 613 of *Contemp. Math.*, pages 41–78. Amer. Math. Soc., Providence, RI, 2014.
- [22] Sergei Gukov, Albert Schwarz, and Cumrun Vafa. Khovanov-Rozansky homology and topological strings. *Lett. Math. Phys.*, 74(1):53–74, 2005.
- [23] Sergei Gukov and Marko Stošić. Homological algebra of knots and BPS states. In *Proceedings of the Freedman Fest*, volume 18 of *Geom. Topol. Monogr.*, pages 309–367. Geom. Topol. Publ., Coventry, 2012.
- [24] Sergei Gukov and Johannes Walcher. Matrix Factorizations and Kauffman Homology. [ArXiv High Energy Physics - Theory e-prints](#), December 2005.

- [25] Magnus Jacobsson. An invariant of link cobordisms from Khovanov homology. *Algebr. Geom. Topol.*, 4:1211–1251 (electronic), 2004.
- [26] Mikhail Khovanov. A categorification of the Jones polynomial. *Duke Math. J.*, 101(3):359–426, 2000.
- [27] Mikhail Khovanov. Triply-graded link homology and Hochschild homology of Soergel bimodules. *Internat. J. Math.*, 18(8):869–885, 2007.
- [28] Mikhail Khovanov and Aaron D. Lauda. A categorification of quantum $\mathfrak{sl}(n)$. *Quantum Topol.*, 1(1):1–92, 2010.
- [29] Mikhail Khovanov and Lev Rozansky. Matrix factorizations and link homology. *Fund. Math.*, 199(1):1–91, 2008.
- [30] Mikhail Khovanov and Lev Rozansky. Matrix factorizations and link homology. II. *Geom. Topol.*, 12(3):1387–1425, 2008.
- [31] Daniel Krasner. Equivariant $\mathfrak{sl}(n)$ -link homology. *Algebr. Geom. Topol.*, 10(1):1–32, 2010.
- [32] P. B. Kronheimer and T. S. Mrowka. Khovanov homology is an unknot-detector. *Publ. Math. Inst. Hautes Études Sci.*, (113):97–208, 2011.
- [33] Lukas Lewark and Andrew Lobb. New Quantum Obstructions to Sliceness. *Proc. London Math. Soc.*, 2015. Accepted.
- [34] Andrew Lobb. A slice genus lower bound from $\mathfrak{sl}(n)$ Khovanov-Rozansky homology. *Adv. Math.*, 222(4):1220–1276, 2009.
- [35] Marco Mackaay, Marko Stošić, and Pedro Vaz. $\mathfrak{sl}(N)$ -link homology ($N \geq 4$) using foams and the Kapustin-Li formula. *Geom. Topol.*, 13(2):1075–1128, 2009.
- [36] Marco Mackaay, Marko Stošić, and Pedro Vaz. The 1, 2-coloured HOMFLY-PT link homology. *Trans. Amer. Math. Soc.*, 363(4):2091–2124, 2011.
- [37] Marco Mackaay and Ben Webster. Categorified skew Howe duality and comparison of knot homologies. *ArXiv e-prints*, February 2015.
- [38] John McCleary. A user’s guide to spectral sequences, volume 58 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2001.
- [39] Victor Mikhaylov and Edward Witten. Branes And Supergroups. *Commun. Math. Phys.*, 340(2):699–832, 2015.
- [40] Hitoshi Murakami, Tomotada Ohtsuki, and Shuji Yamada. Homfly polynomial via an invariant of colored plane graphs. *Enseign. Math.* (2), 44(3-4):325–360, 1998.
- [41] Peter Ozsváth and Zoltán Szabó. Holomorphic disks, link invariants and the multi-variable Alexander polynomial. *Algebr. Geom. Topol.*, 8(2):615–692, 2008.
- [42] Hoel Queffelec and David E. V. Rose. The \mathfrak{sl}_n foam 2-category: a combinatorial formulation of khovanov-rozansky homology via categorical skew howe duality. *ArXiv e-prints*, May 2014.
- [43] Hoel Queffelec and David E. V. Rose. Sutured annular Khovanov-Rozansky homology. *ArXiv e-prints*, June 2015.
- [44] J. Rasmussen. Floer homology and knot complements. PhD thesis, Harvard University, June 2003.
- [45] Jacob Rasmussen. Khovanov-Rozansky homology of two-bridge knots and links. *Duke Math. J.*, 136(3):551–583, 2007.
- [46] Jacob Rasmussen. Khovanov homology and the slice genus. *Invent. Math.*, 182(2):419–447, 2010.
- [47] Jacob Rasmussen. Some differentials on Khovanov–Rozansky homology. *Geom. Topol.*, 19(6):3031–3104, 2015.
- [48] Nicolai Yu. Reshetikhin and Vladimir G. Turaev. Ribbon graphs and their invariants derived from quantum groups. *Comm. Math. Phys.*, 127(1):1–26, 1990.
- [49] David E. V. Rose and Paul Wedrich. Deformations of colored $\mathfrak{sl}(N)$ link homologies via foams. *Geom. Topol.*, 2015. Accepted.
- [50] Lev Rozansky. An infinite torus braid yields a categorified Jones-Wenzl projector. *Fund. Math.*, 225(1):305–326, 2014.
- [51] D. Tubbenhauer, P. Vaz, and P. Wedrich. Super q -Howe duality and web categories. *ArXiv e-prints*, April 2015.
- [52] Ben Webster. Khovanov-Rozansky homology via a canopolis formalism. *Algebr. Geom. Topol.*, 7:673–699, 2007.
- [53] Ben Webster and Geordie Williamson. A geometric construction of colored HOMFLYPT homology. *ArXiv e-prints*, May 2009.
- [54] Paul Wedrich. Categorified $\mathfrak{sl}(N)$ invariants of colored rational tangles. *Algebr. Geom. Topol.* Accepted.
- [55] Geordie Williamson. Singular Soergel bimodules. *Int. Math. Res. Not. IMRN*, (20):4555–4632, 2011.
- [56] Hao Wu. On the quantum filtration of the Khovanov-Rozansky cohomology. *Adv. Math.*, 221(1):54–139, 2009.
- [57] Hao Wu. Generic deformations of the colored $\mathfrak{sl}(N)$ -homology for links. *Algebr. Geom. Topol.*, 11(4):2037–2106, 2011.
- [58] Hao Wu. Equivariant colored $\mathfrak{sl}(N)$ -homology for links. *J. Knot Theory Ramifications*, 21(2):1250012, 104, 2012.
- [59] Hao Wu. A colored $\mathfrak{sl}(N)$ homology for links in S^3 . *Dissertationes Math. (Rozprawy Mat.)*, 499:217, 2014.
- [60] Yasuyoshi Yonezawa. Quantum $(\mathfrak{sl}_n, \wedge V_n)$ link invariant and matrix factorizations. *Nagoya Math. J.*, 204:69–123, 2011.